5.163 global_cardinality

DESCRIPTION LINKS GRAPH AUTOMATON

Origin

CHARME [298]

Constraint

global_cardinality(VARIABLES, VALUES)

Synonyms

count, distribute, distribution, gcc, card_var_gcc, egcc, extended_global_cardinality.

Arguments

VARIABLES : collection(var−dvar)
VALUES : collection(val−int, noccurrence−dvar)

Restrictions

required(VARIABLES, var)
required(VALUES, [val, noccurrence])
distinct(VALUES, val)
VALUES.noccurrence ≥ 0
VALUES.noccurrence $\leq$ |VARIABLES|

Purpose

Each value VALUES[i].val (with $i \in [1, |\texttt{VALUES}|]$) should be taken by exactly VALUES[i].noccurrence variables of the VARIABLES collection.

Example

$$\left(\begin{array}{l}\langle 3,3,8,6\rangle, \\ \left\langle\begin{array}{ll}\texttt{val}-3 & \texttt{noccurrence}-2, \\ \texttt{val}-5 & \texttt{noccurrence}-0, \\ \texttt{val}-6 & \texttt{noccurrence}-1\end{array}\right\rangle\end{array}\right)$$

The global_cardinality constraint holds since values 3, 5 and 6 respectively occur 2, 0 and 1 times within the collection $\langle 3,3,8,6\rangle$ and since no restriction was specified for value 8.

All solutions

Figure 5.352 gives all solutions to the following non ground instance of the global_cardinality constraint: $V_1 \in [3,4]$, $V_2 \in [2,3]$, $V_3 \in [1,2]$, $V_4 \in [2,4]$, $V_5 \in [2,3]$, $V_6 \in [1,2]$, $O_1 \in [1,1]$, $O_2 \in [2,3]$, $O_3 \in [0,1]$, $O_4 \in [2,3]$, global_cardinality($\langle V_1, V_2, V_3, V_4, V_5, V_6\rangle, \langle 1\, O_1,\ 2\, O_2,\ 3\, O_3,\ 4\, O_4\rangle$).

Typical

|VARIABLES| > 1
range(VARIABLES.var) > 1
|VALUES| > 1
|VARIABLES| $\geq$ |VALUES|
minval(VARIABLES.var) $= 0 \vee$ in_attr(VARIABLES, var, VALUES, val)

① $(\langle 4, 2, 1, 4, 2, 2\rangle, \langle 1\ \mathbf{1},\ 2\ \mathbf{3},\ 3\ \mathbf{0},\ 4\ \mathbf{2}\rangle)$
② $(\langle 4, 2, 1, 4, 3, 2\rangle, \langle 1\ \mathbf{1},\ 2\ \mathbf{2},\ 3\ \mathbf{1},\ 4\ \mathbf{2}\rangle)$
③ $(\langle 4, 2, 2, 4, 2, 1\rangle, \langle 1\ \mathbf{1},\ 2\ \mathbf{3},\ 3\ \mathbf{0},\ 4\ \mathbf{2}\rangle)$
④ $(\langle 4, 2, 2, 4, 3, 1\rangle, \langle 1\ \mathbf{1},\ 2\ \mathbf{2},\ 3\ \mathbf{1},\ 4\ \mathbf{2}\rangle)$
⑤ $(\langle 4, 3, 1, 4, 2, 2\rangle, \langle 1\ \mathbf{1},\ 2\ \mathbf{2},\ 3\ \mathbf{1},\ 4\ \mathbf{2}\rangle)$
⑥ $(\langle 4, 3, 2, 4, 2, 1\rangle, \langle 1\ \mathbf{1},\ 2\ \mathbf{2},\ 3\ \mathbf{1},\ 4\ \mathbf{2}\rangle)$

Figure 5.352: All solutions corresponding to the non ground example of the `global_cardinality` constraint of the **All solutions** slot

Symmetries

- Items of `VARIABLES` are permutable.
- Items of `VALUES` are permutable.
- An occurrence of a value of `VARIABLES.var` that does not belong to `VALUES.val` can be replaced by any other value that also does not belong to `VALUES.val`.
- All occurrences of two distinct values in `VARIABLES.var` or `VALUES.val` can be swapped; all occurrences of a value in `VARIABLES.var` or `VALUES.val` can be renamed to any unused value.

Arg. properties

- Functional dependency: `VALUES.noccurrence` determined by `VARIABLES` and `VALUES.val`.
- Contractible wrt. `VALUES`.

Usage

We show how to use the `global_cardinality` constraint in order to model the magic series problem [415, page 155] with a single `global_cardinality` constraint. A non-empty finite series $S = (s_0, s_1, \ldots, s_n)$ is *magic* if and only if there are s_i occurrences of i in S for each integer i ranging from 0 to n. This leads to the following model:

$$\texttt{global\_cardinality}\left(\begin{array}{l}\langle\ \texttt{var}-s_0, \texttt{var}-s_1, \ldots, \texttt{var}-s_n\ \rangle,\\ \left\langle\begin{array}{ll}\texttt{val}-0 & \texttt{noccurrence}-s_0,\\ \texttt{val}-1 & \texttt{noccurrence}-s_1,\\ & \vdots\\ \texttt{val}-n & \texttt{noccurrence}-s_n\end{array}\right\rangle\end{array}\right)$$

Remark

This is a generalised form of the original `global_cardinality` constraint: in the original `global_cardinality` constraint [342], one specifies for each value its minimum and maximum number of occurrences (i.e., see `global_cardinality_low_up`). Here we give for each value v a domain variable that indicates how many time value v is actually used. By setting the minimum and maximum values of this variable to the appropriate constants we can express the same thing as in the original `global_cardinality` constraint. However, as shown in the *magic series* problem, we can also use this variable in other constraints. By reduction from 3-SAT, Claude-Guy Quimper shows in [331] that it is NP-hard to achieve arc-consistency for the count variables.

A last difference with the original global_cardinality constraint comes from the fact that there is no constraint on the values that are not explicitly mentioned in the VALUES collection. In the original global_cardinality these values could not be assigned to the variables of the VARIABLES collection. However allowing values that are not mentioned in VALUES to be assigned to variables of VARIABLES can potentially avoid mentioning a huge number of unconstrained values in the VALUES collection, and as a side effect, prevent possibly[6] generating a dense graph (i.e., see DFS-bottleneck) for the corresponding underlying flow model).

Within [83] the global_cardinality constraint is called distribution. Within [350] the global_cardinality constraint is called card_var_gcc. Within [70] the global_cardinality constraint is called egcc or rgcc. This later case corresponds to the fact that some variables are duplicated within the VARIABLES collection.

The global_cardinality constraint can be seen as a system (i.e., a conjunction) of among constraints.

When all count variables (i.e., the variables $\texttt{VALUES}[i].\texttt{noccurrence}$ with $i \in [1, |\texttt{VALUES}|]$) *do not occur* in any other constraints of the problem, it may be operationally more efficient to replace the global_cardinality constraint by a global_cardinality_low_up constraint where each count variable $\texttt{VALUES}[i].\texttt{noccurrence}$ is replaced by the corresponding interval $[\underline{\texttt{VALUES}[i].\texttt{noccurrence}}, \overline{\texttt{VALUES}[i].\texttt{noccurrence}}]$. This stands for two reasons:

- First, by using a global_cardinality_low_up constraint rather than a global_cardinality constraint, we avoid the filtering algorithm related to the count variables.
- Second, unlike the global_cardinality constraint where we need to fix all its variables to get entailment, the global_cardinality_low_up constraint can be entailed before all its variables get fixed. As a result, this potentially avoid unnecessary calls to its filtering algorithm.

When all values that can be assigned to the variables of the VARIABLES collection occur in the val attribute of the VALUES collection, two implicit *necessary* conditions[7] inferred by double counting with the global_cardinality constraint are depicted by the following expressions:

$$|\texttt{VARIABLES}| = \sum_{i=1}^{|\texttt{VALUES}|} \texttt{VALUES}[i].\texttt{noccurrence}$$

$$\sum_{i=1}^{|\texttt{VARIABLES}|} \texttt{VARIABLES}[i].\texttt{var} = \sum_{i=1}^{|\texttt{VALUES}|} \texttt{VALUES}[i].\texttt{val} \cdot \texttt{VALUES}[i].\texttt{noccurrence}$$

Within [317, pages 50–51] the previous condition where terms involving identical variables are grouped together (i.e., rule 5 of MALICE [316]) is mentioned as a crucial deduction rule for the autoref problem.

[6] Of course one could also, while generating a flow model, detect all unconstrained values in order to generate a single vertex in the flow model for the set of unconstrained values.

[7] Note that such necessary conditions can be derived by assigning an integer weight to each value [385], e.g. 1 for the first condition, the value itself for the second condition.

W.-J. van Hoeve *et al.* present two soft versions of the global_cardinality constraint in [424].

In **MiniZinc** (http://www.minizinc.org/) there is also a distribute constraint where the val attribute is not necessarily initially fixed and where a same value may occur more than once. Their is also a global_cardinality_closed constraint where all variables must be assigned a value from the val attribute.

Algorithm A flow algorithm that handles the original global_cardinality constraint is described in [342]. The two approaches that were used to design bound-consistency algorithms for alldifferent were generalised for the global_cardinality constraint. The algorithm in [334] identifies Hall intervals and the one in [233] exploits convexity to achieve a fast implementation of the flow-based arc-consistency algorithm. The later algorithm can also compute bound-consistency for the count variables [234, 231]. An improved algorithm for achieving arc-consistency is described in [333].

Systems globalCardinality in **Choco**, count in **Gecode**, gcc in **JaCoP**, global_cardinality in **MiniZinc**, global_cardinality in **SICStus**.

See also

common keyword: count, max_nvalue, min_nvalue *(value constraint,counting constraint)*, nvalue *(counting constraint)*, open_global_cardinality_low_up *(assignment,counting constraint)*.

cost variant: global_cardinality_with_costs *(*cost *associated with each* variable*,*value *pair)*.

implied by: global_cardinality_with_costs *(forget about cost)*, same_and_global_cardinality *(conjoin* same *and* global_cardinality*)*.

part of system of constraints: among.

related: roots, sliding_card_skip0 *(counting constraint of a set of values on maximal sequences)*.

shift of concept: global_cardinality_no_loop *(assignment of a* variable *to its position is ignored)*, ordered_global_cardinality *(restrictions are done on nested sets of values, all starting from first value)*, symmetric_cardinality, symmetric_gcc.

soft variant: open_global_cardinality *(a* set variable *defines the set of variables that are actually considered)*.

specialisation: alldifferent *(each value should occur at most once)*, cardinality_atleast, cardinality_atmost *(individual* count variable *for each value replaced by single* count variable*)*, cardinality_atmost_partition *(individual* count variable *for each value replaced by single* count variable *and* variable $\in$ partition *replaced by* variable*)*, global_cardinality_low_up *(*variable *replaced by* fixed interval*)*.

system of constraints: colored_matrix *(one* global_cardinality *constraint for each* row *and each* column *of a* matrix *of* variables*)*.

uses in its reformulation: tree_range, tree_resource.

Keywords

application area: assignment.

characteristic of a constraint: core, automaton, automaton with array of counters.

complexity: 3-SAT.

constraint arguments: pure functional dependency.

constraint type: value constraint, counting constraint, system of constraints.

filtering: Hall interval, bound-consistency, flow, duplicated variables, DFS-bottleneck.

modelling: functional dependency.

modelling exercises: magic series.

puzzles: magic series, autoref.

Cond. implications

- global_cardinality(VARIABLES, VALUES)
  with minval(VARIABLES.var) $= 0$
  implies and(VAR, VARIABLES)
  when VAR $= 0$.

- global_cardinality(VARIABLES, VALUES)
  with maxval(VARIABLES.var) $= 1$
  implies or(VAR, VARIABLES)
  when VAR $= 1$.

- global_cardinality(VARIABLES, VALUES)
  with minval(VARIABLES.var) > 0
  implies min_size_full_zero_stretch(MINSIZE, VARIABLES)
  when MINSIZE $=$ |VARIABLES|.

- global_cardinality(VARIABLES, VALUES)
  with maxval(VARIABLES.var) < 0
  implies min_size_full_zero_stretch(MINSIZE, VARIABLES)
  when MINSIZE $=$ |VARIABLES|.

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) $=$nval(VALUES.val)
  and minval(VALUES.val) $\leq$minval(VARIABLES.var)
  and maxval(VALUES.val) $\geq$maxval(VARIABLES.var)
  implies among_diff_0(NVAR, VARIABLES).

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) $=$nval(VALUES.val)
  and minval(VALUES.val) $\leq$minval(VARIABLES.var)
  and maxval(VALUES.val) $\geq$maxval(VARIABLES.var)
  implies atmost_nvalue(NVAL, VARIABLES).

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.noccurrence) $= 1$
  and range(VALUES.val) $=$nval(VALUES.val)
  and minval(VALUES.val) $=$minval(VARIABLES.var)
  and maxval(VALUES.val) $=$maxval(VARIABLES.var)
  implies balance(BALANCE, VARIABLES)
  when BALANCE $= 0$.

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) $=$nval(VALUES.val)
  and minval(VALUES.val) $\leq$minval(VARIABLES.var)
  and maxval(VALUES.val) $\geq$maxval(VARIABLES.var)
  implies max_n(MAX, RANK, VARIABLES).

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) =nval(VALUES.val)
  and minval(VALUES.val) ≤minval(VARIABLES.var)
  and maxval(VALUES.val) ≥maxval(VARIABLES.var)
  implies max_nvalue(MAX, VARIABLES).

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) =nval(VALUES.val)
  and minval(VALUES.val) ≤minval(VARIABLES.var)
  and maxval(VALUES.val) ≥maxval(VARIABLES.var)
  implies min_n(MIN, RANK, VARIABLES).

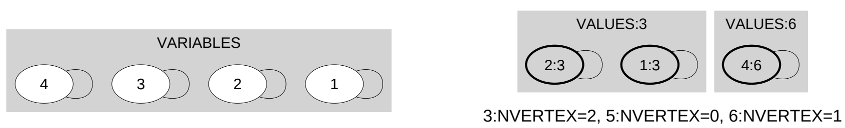
- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) =nval(VALUES.val)
  and minval(VALUES.val) ≤minval(VARIABLES.var)
  and maxval(VALUES.val) ≥maxval(VARIABLES.var)
  implies min_nvalue(MIN, VARIABLES).

- global_cardinality(VARIABLES, VALUES)
  with range(VALUES.val) =nval(VALUES.val)
  and minval(VALUES.val) ≤minval(VARIABLES.var)
  and maxval(VALUES.val) ≥maxval(VARIABLES.var)
  implies range_ctr(VARIABLES, CTR, R).

For all items of VALUES:

Arc input(s)	VARIABLES
Arc generator	$SELF \mapsto$ collection(variables)
Arc arity	1
Arc constraint(s)	variables.var = VALUES.val
Graph property(ies)	**NVERTEX** = VALUES.noccurrence

Graph model Since we want to express one unary constraint for each value we use the "For all items of VALUES" iterator. Part (A) of Figure 5.353 shows the initial graphs associated with each value 3, 5 and 6 of the VALUES collection of the **Example** slot. Part (B) of Figure 5.353 shows the two corresponding final graphs respectively associated with values 3 and 6 that are both assigned to the variables of the VARIABLES collection (since value 5 is not assigned to any variable of the VARIABLES collection the final graph associated with value 5 is empty). Since we use the **NVERTEX** graph property, the vertices of the final graphs are stressed in bold.



(A) **(B)**

Figure 5.353: Initial and final graph of the global_cardinality constraint

Automaton Figure 5.354 depicts the automaton associated with the `global_cardinality` constraint. To each item of the collection `VARIABLES` corresponds a signature variable S_i that is equal to 0. To each item of the collection `VALUES` corresponds a signature variable $S_{i+|\mathtt{VARIABLES}|}$ that is equal to 1.

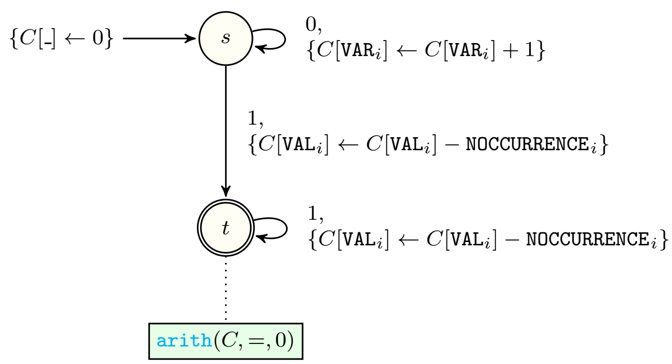


Figure 5.354: Automaton of the `global_cardinality` constraint

Quiz

EXERCISE 1 (checking whether a ground instance holds or not)[a]

A. Does the constraint
$\mathtt{global\_cardinality}(\langle 2,4,2,2,1\rangle, \langle 0\ 0, 1\ 1, 2\ 3, 3\ 0, 4\ 1\rangle)$ hold?

B. Does the constraint
$\mathtt{global\_cardinality}(\langle 0,0,1,1\rangle, \langle 0\ 2, 1\ 2, 2\ 1, 3\ 0, 4\ 0\rangle)$ hold?

C. Does the constraint
$\mathtt{global\_cardinality}(\langle 2,3,4,5\rangle, \langle 0\ 0, 1\ 0, 2\ 1, 3\ 1, 4\ 1\rangle)$ hold?

[a] Hint: go back to the definition of `global_cardinality`.

EXERCISE 2 (finding all solutions)[a]

Give all the solutions to the constraint:

$$\left\{\begin{array}{lll} V_1 \in [1,2], & V_2 \in [1,2], & V_3 \in [1,2], \\ V_4 \in [2,3], & V_5 \in [3,3], & \\ O_1 \in [1,2], & O_2 \in [2,3], & O_3 \in [0,1], \\ \multicolumn{3}{l}{\mathtt{global\_cardinality}\left(\begin{array}{l} \langle V_1,V_2,V_3,V_4,V_5\rangle, \\ \left\langle \begin{array}{ll} \mathtt{val}-1 & \mathtt{occurrence}-O_1, \\ \mathtt{val}-2 & \mathtt{occurrence}-O_2, \\ \mathtt{val}-3 & \mathtt{occurrence}-O_3 \end{array}\right\rangle \end{array}\right).} \end{array}\right.$$

[a] Hint: focus on the variables of the first argument (since the counting variables of the second argument are functionally determined by the first argument), and enumerate solutions in lexicographic order.

EXERCISE 3 (identifying infeasible values)[a]

Identify all variable-value pairs (V_i, val) (respectively (O_i, val)) (with $i \in [1,5]$), such that the following constraint has no solution when variable V_i (respectively O_i) is assigned value val:

$$\left\{\begin{array}{lll} V_1 \in [2,3], & V_2 \in [1,5], & V_3 \in [3,4], \\ V_4 \in [1,3], & V_5 \in [1,4], & \\ O_1 \in [1,4], & O_2 \in [0,1], & O_3 \in [0,1], \\ O_4 \in [1,5], & O_5 \in [1,4], & \\ \multicolumn{3}{l}{\texttt{global\_cardinality}\left(\begin{array}{l} \langle V_1, V_2, V_3, V_4, V_5 \rangle, \\ \left\langle \begin{array}{ll} \texttt{val}-1 & \texttt{occurrence}-O_1, \\ \texttt{val}-2 & \texttt{occurrence}-O_2, \\ \texttt{val}-3 & \texttt{occurrence}-O_3, \\ \texttt{val}-4 & \texttt{occurrence}-O_4, \\ \texttt{val}-5 & \texttt{occurrence}-O_5 \end{array} \right\rangle \end{array}\right)} \end{array}\right. .$$

[a] Hint: first restrict the occurrence variables $O_1, O_2, \ldots, O_5$, second restrict the decision variables $V_1, V_2, \ldots, V_5$, third check that all remaining values occur in at least one solution.

EXERCISE 4 (modelling a nurse assignment problem)[a]

Given a 24 hour period, you must schedule a pool of six nurses Bea, Lea, Leo, Lio , Lili and $\mathrm{Tom}$to *at least two* and *at most three morning shifts*, to *at least two* and *at most three afternoon shifts*, to *at least one night shift*, while the other nurses are off-duty. In addition, due to past work, we have the following extra requirements:

- Since on the previous 24 hour period Bea, Lea and Leo were working in the afternoon shift they cannot be assigned to the night shift.
- Leo, Lio and Lili have to work since they already took all their days off.
- Bea and Tomhave to work together since Bea supervises Tom

Provide a model of this problem that uses the `global_cardinality` constraint.

A. Provide a solution that satisfies all the constraints, i.e., for each nurse give his/her assignment ($\mathrm{morning}$, $\mathrm{afternoon}$, night , $\mathrm{off\text{-}duty}$).

B. Identify the decision variables and the values of the problem, i.e., how do we model the fact that nurse $x \in \{\mathrm{Bea}, \mathrm{Lea}, \mathrm{Leo}, \mathrm{Lio}, \mathrm{Lili}, \mathrm{Tom}\}$ is assigned shift $y \in \{\mathrm{morning}, \mathrm{afternoon}, \mathrm{night}, \mathrm{off\text{-}duty}\}$?

C. Using a bipartite graph, draw the relations between the variables and the values identified in the previous question and display the solution you came up with in the first question.

D. Provide a model of the problem that uses a single `global_cardinality` constraint.

- Explain how the minimum/maximum capacity constraints (i.e., at least/at most) are modelled.
- Explain how each extra requirement is modelled in your solution.

[a] Hint: focus on *what is a variable* and *what is a value* in your model, and how to model the capacity constraints with `global_cardinality`.

SOLUTION TO EXERCISE 1

A. *Yes, since within* $\langle 2, 4, 2, 2, 1\rangle$, *values* 0, 1, 2, 3 *and* 4 *are respectively used zero, one, three, zero, and one times.*

B. *No, since within* $\langle 0, 0, 1, 1\rangle$, *value* 2 *is not used one time.*

C. *Yes, since within* $\langle 2, 3, 4, 5\rangle$, *value* 0, 1, 2, 3 *and* 4 *are respectively used zero, zero, one, one, and one times. The presence of a* 5 *in the solution does not matter since value* 5 *is not mentioned in the values of the second argument of the* `global_cardinality` *constraint.*

SOLUTION TO EXERCISE 2

the six solutions

$\langle V_1, V_2, V_3, V_4, V_5\rangle$ $\langle 1\ O_1, 2\ O_2, 3\ O_3\rangle$

① $(\langle 1, 1, 2, 2, 3\rangle,\ \langle 1\ \mathbf{2}, 2\ \mathbf{2}, 3\ \mathbf{1}\rangle)$

② $(\langle 1, 2, 1, 2, 3\rangle,\ \langle 1\ \mathbf{2}, 2\ \mathbf{2}, 3\ \mathbf{1}\rangle)$

③ $(\langle 1, 2, 2, 2, 3\rangle,\ \langle 1\ \mathbf{1}, 2\ \mathbf{3}, 3\ \mathbf{1}\rangle)$

④ $(\langle 2, 1, 1, 2, 3\rangle,\ \langle 1\ \mathbf{2}, 2\ \mathbf{2}, 3\ \mathbf{1}\rangle)$

⑤ $(\langle 2, 1, 2, 2, 3\rangle,\ \langle 1\ \mathbf{1}, 2\ \mathbf{3}, 3\ \mathbf{1}\rangle)$

⑥ $(\langle 2, 2, 1, 2, 3\rangle,\ \langle 1\ \mathbf{1}, 2\ \mathbf{3}, 3\ \mathbf{1}\rangle)$

SOLUTION TO EXERCISE 3

As suggested by the hint we go through the following steps:

A. [*RESTRICTING THE OCCURRENCE VARIABLES* $O_1, O_2, \ldots, O_5$]

(a) [*PRUNING WRT THE MAXIMUM NUMBER OF OCCURRENCES OF EACH VALUE*]
Since values 1, 2, 3, 4 *and* 5 *can respectively be assigned to at most* 3, 4, 5, 3 *and* 1 *decision variables (e.g., value* 1 *can only be assigned to* V_2, V_4 *and* V_5*) we have* $O_1 \leq \min(3,4)$, $O_2 \leq \min(4,1)$, $O_3 \leq \min(5,1)$, $O_4 \leq \min(3,5)$, *and* $O_5 \leq \min(1,4)$.

(b) [*PRUNING WRT* $\sum_{i=1}^{5} O_i = 5$ *AND THE DOMAIN OF* V_1]
Since we have five decision variables the sum of the occurrence variables is equal to five (i.e., $O_1 + O_2 + O_3 + O_4 + O_5 = 5$*). Since values* 2 *or* 3 *have to be assigned to the decision variable* V_1 *we have* $O_2 + O_3 \geq 1$. *It follows that* $O_1 + O_4 + O_5 \leq 4$. *Since* $O_1 \in [1,3]$, $O_4 \in [1,3]$ *and* $O_5 = 1$ *we get* $O_1 + O_4 \leq 3$ *and consequently* $O_1 \leq 2$ *and* $O_4 \leq 2$.

B. [*RESTRICTING THE DECISION VARIABLES* $V_1, V_2, \ldots, V_5$]
At the end of step **A** *we obtain* $O_1 \in [1,2]$, $O_2 \in [0,1]$, $O_3 \in [0,1]$, $O_4 \in [1,2]$, *and* $O_5 \in [1,1]$. *Since* $O_5 = 1$ *and since* V_2 *is the only decision variable that can be assigned value* 5 *we have* $V_2 = 5$. *Consequently* $V_1 \in [2,3]$, $V_2 \in [5,5]$, $V_3 \in [3,4]$, $V_4 \in [1,3]$, *and* $V_5 \in [1,4]$.

C. [*CHECKING FOR A SUPPORT*]
To show that no value can be removed from the domain of the decision and occurrence variables we show that every value that is still in the domain of a variable is part of a solution.

(a) *A solution with* $O_1 = 2$ *is* $V_1 = 2$, $V_2 = 5$, $V_3 = 4$, $V_4 = 1$, $V_5 = 1$ *and* $O_1 = 2$, $O_2 = 1$, $O_3 = 0$, $O_4 = 1$, $O_5 = 1$.

(b) *A solution with* $O_4 = 2$ *is* $V_1 = 2$, $V_2 = 5$, $V_3 = 4$, $V_4 = 1$, $V_5 = 4$ *and* $O_1 = 1$, $O_2 = 1$, $O_3 = 0$, $O_4 = 2$, $O_5 = 1$.

(c) *We now assume that* $O_1 = O_2 = O_3 = O_4 = O_5 = 1$, *i.e., all decision variables must be distinct. Without loos of generality we ignore* V_2, *which is fixed to* 5. *We provide a set of solutions where* V_1, V_3, V_4 *and* V_5 *can respectively be assigned to all the values of their domains:*

i. $V_1 = 2,\ \ V_3 = 3,\ \ V_4 = 1,\ \ V_5 = 4,$
ii. $V_1 = 2,\ \ V_3 = 4,\ \ V_4 = 3,\ \ V_5 = 1,$
iii. $V_1 = 2,\ \ V_3 = 4,\ \ V_4 = 1,\ \ V_5 = 3,$
iv. $V_1 = 3,\ \ V_3 = 4,\ \ V_4 = 1,\ \ V_5 = 2,$
v. $V_1 = 3,\ \ V_3 = 4,\ \ V_4 = 2,\ \ V_5 = 1.$

SOLUTION TO EXERCISE 4

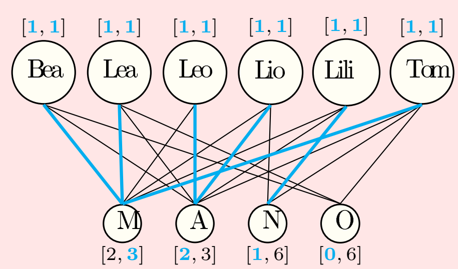
A. *A feasible solution is the following assignment*

$$\left(\begin{array}{lll} \mathrm{Bea}:\mathrm{morning} & \mathrm{Lea}:\mathrm{morning} & \mathrm{Leo}:\mathrm{afternoon} \\ \mathrm{Lio}:\mathrm{afternoon} & \mathrm{Lili}:\mathrm{night} & \mathrm{Tom}:\mathrm{morning} \end{array}\right) \textit{ since:}$$

- *the number of morning shifts is between* 2 *and* 3,
- *the number of afternoon shifts is between* 2 *and* 3,
- *the number of night shifts is at least* 1,
- Bea, Lea *and* Leo *are not assigned to a night shift,*
- Leo, Lio *and* Lili *work,*
- Bea *and* Tom *are both assigned the same shift.*

B. *To each nurse corresponds a variable whose initial domain is set to the types of shifts that nurse can actually perform (i.e., each shift type is encoded by a unique integer value).*

C. *The next figure provides a graphical representation of the assignment problem. To each nurse and to each shift type corresponds a vertex. There is an edge between a given nurse and a given shift type if and only if that nurse can perform that shift type. The solution given to question* **A** *is displayed with thick blue lines. The interval on top or below each vertex indicates the minimum and maximum number of edges that can reach the corresponding vertex in any solution; values in blue correspond to the number of edges of the displayed solution.*



D. *We get the following model*

$$\left\{\begin{array}{llll} \mathrm{M}=1, & \mathrm{A}=2, & \mathrm{N}=3, & \mathrm{O}=4, \\ \mathrm{Bea}\in[\mathrm{M},\mathrm{O}], & \mathrm{Lea}\in[\mathrm{M},\mathrm{O}], & \mathrm{Leo}\in[\mathrm{M},\mathrm{O}], & \\ \mathrm{Lio}\in[\mathrm{M},\mathrm{O}], & \mathrm{Lili}\in[\mathrm{M},\mathrm{O}], & \mathrm{Tom}\in[\mathrm{M},\mathrm{O}], & \\ O\mathrm{M}\in[2,3], & O\mathrm{A}\in[2,3], & O\mathrm{N}\in[1,6], & O\mathrm{O}\in[0,6], \\ \mathrm{Bea}\neq\mathrm{N}, & \mathrm{Lea}\neq\mathrm{N}, & \mathrm{Leo}\neq\mathrm{N}, & \\ \mathrm{Leo}\neq\mathrm{O}, & \mathrm{Lio}\neq\mathrm{O}, & \mathrm{Lili}\neq\mathrm{O}, & \\ \mathrm{Bea}=\mathrm{Tom}, & & & \\ \multicolumn{4}{l}{\texttt{global\_cardinality}\left(\begin{array}{c}\langle\mathrm{Bea},\mathrm{Lea},\mathrm{Leo},\mathrm{Lio},\mathrm{Lili},\mathrm{Tom}\rangle, \\ \left\langle\begin{array}{ll}\texttt{val}-\mathrm{M} & \texttt{occurrence}-O\mathrm{M}, \\ \texttt{val}-\mathrm{A} & \texttt{occurrence}-O\mathrm{A}, \\ \texttt{val}-\mathrm{N} & \texttt{occurrence}-O\mathrm{N}, \\ \texttt{val}-\mathrm{O} & \texttt{occurrence}-O\mathrm{O}\end{array}\right\rangle\end{array}\right),} \end{array}\right.$$

where:

- *line 1 declares the integer value of each shift type,*
- *lines 2, 3 and 4 declare the nurse and occurrence variables,*
- *line 5 enforces* Bea, Lea *and* Leo *not to work on a night shift,*
- *line 6 imposes* Leo, Lio *and* Lili *to work,*
- *line 7 constrains* Bea *and* Tom *to work on the same shift,*
- *line 8 restricts each shift type to occur within a given range.*