M2 ORO: Advanced Integer Programming
Solution Final Exam – 1st session

november 16, 2009

duration: 2 hours.

documents: lecture notes are authorized. No book, no book copy.

grades: 4 problems of 5 points each = 20 points.

Notations:

$\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_{*+}$	the sets of integer, non-negative integer, and positive integer numbers
$\mathbb{R}, \mathbb{R}_+, \mathbb{R}_{*+}$	the sets of real, non-negative real, and positive real numbers
$\lceil a \rceil, \forall a \in \mathbb{R}$	the integer round-up (ceiling) of a: $\min\{b \in \mathbb{Z} \mid a \leqslant b\}$
$\lfloor a \rfloor, \forall a \in \mathbb{R}$	the integer round-down (floor) of a: $\max\{b \in \mathbb{Z} \mid a \geqslant b\}$

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1 Multi-Item Lot-Sizing Problem (MLS)

Problem 1 The Multi-Item Lot-Sizing Problem (MLS).
Given demands d_t^k for items $k = 1, \ldots, K$ over a time horizon $t = 1, \ldots, T$. All items must be produced on a single machine. The machine has to produce exactly one item type in each period. Furthermore, the machine has no capacity: it can produce any number of items in a given period. Given unit production costs p_t^k, unit storage costs h_t^k and set-up costs f_t^k for each item k in each period t, we wish to find a minimum cost production plan.

Question 1 ***(5 points).***

Q1.1. model this problem (MLS) as a Mixed Integer Linear Program;

Q1.2. suppose that we can compute all the feasible production plans for each item k individually, i.e. all the feasible solutions of the Uncapacitated Lot-Sizing Problem induced for each item k. Let X^k denote this set of feasible solutions and n^k the cardinality of X^k. Show that (MLS) can be seen as an instance of the Set-Partitioning Problem and derive a second model for (MLS) as a Binary Integer Linear Program with $\sum_{k=1}^K n^k$ binary variables.

1.

$$
\begin{aligned}
(MLS): z \;=\; \min & \sum_k \sum_t p_t^k x_t^k + h_t^k s_t^k + f_t^k y_t^k \\
\text{s.t.} & \sum_k y_t^k = 1 && \forall t = 1, \ldots, T, \\
& (x^k, s^k, y^k) \in X^k && \forall k = 1, \ldots, K,
\end{aligned}
$$

where

$$
\begin{aligned}
X^k = \{\, & (x^k, s^k, y^k) \in \mathbb{R}_+^T \times \mathbb{R}_+^{T+1} \times \{0,1\}^T \mid \\
& s_{t-1}^k + x_t^k = d_t^k + s_t^k && \forall t = 1, \ldots, T, \\
& x_t^k \leqslant M^k y_t^k && \forall t = 1, \ldots, T, \\
& s_0^k = 0 && \}
\end{aligned}
$$

is the set of feasible solutions of an Uncapacitated Lot-Sizing Problem.

2. (MLS) is an instance of set-partitioning where one must select exactly one feasible production plan for each item type (i.e. one element of X^k for each $k = 1, \ldots, K$) such that exactly one item is produced at each time $t = 1, \ldots, T$. Let $P^{k,1}, \ldots, P^{k,n^k}$ denote the elements of X^k, i.e: each $P^{k,i}$ corresponds to a vector $(x^{k,i}, s^{k,i}, y^{k,i})$ satisfying the constraints of X^k. Let $a_t^{k,i} = y_t^{k,i}$, $c^{k,i} = \sum_t p_t^k x_t^{k,i} + h_t^k s_t^{k,i} + f_t^k y_t^{k,i}$. Then consider a decision variable $\lambda^{k,i} = 1$ if and only if the production plan $P^{k,i}$ is selected. Then (MLS) can be reformulated as the following set-partitioning problem:

$$
\begin{aligned}
(MLS'): z \;=\; \min & \sum_k \sum_{i=1}^{n^k} c^{k,i} \lambda^{k,i} \\
\text{s.t.} & \sum_k \sum_{i=1}^{n^k} a_t^{k,i} \lambda^{k,i} = 1 && \forall t = 1, \ldots, T, \\
& \sum_{i=1}^{n^k} \lambda^{k,i} = 1 && \forall k = 1, \ldots, K, \\
& \lambda^{k,i} \in \{0,1\} && \forall k = 1, \ldots, K, \forall i = 1, \ldots, n^k,
\end{aligned}
$$

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2 Traveling Salesman Problem with Time Windows (TSP-TW)

Problem 2 The Traveling Salesman Problem with Time Windows (TSP-TW).
*The TSPTW is defined on a network $G = (N \cup \{0\}, A)$ where $N = \{1, \ldots, n\}$ is the set of nodes to visit, 0 is the depot, and A is the set of arcs connecting each pairs of distinct nodes. To each arc $(i,j) \in A$, are associated a cost $c_{ij} \geqslant 0$ and a travel duration $t_{ij} > 0$. To each node $i \in N \cup \{0\}$, is associated a time window $[a_i, b_i]$, with $0 \leqslant a_i \leqslant b_i$. A **tour** is a path starting at the depot at time 0, visiting all nodes in N exactly once, then returning to the depot. Given a tour, we can associate an **arrival time** to each node $i \in N \cup \{0\}$ such that: if arc $(0,i)$ belongs to the tour then the arrival time at i is greater or equal to the travel duration t_{0i}; and if arc (i,j), $i \neq 0$, belongs to the tour then the arrival time at j is greater or equal to the arrival time at i plus the travel duration t_{ij}. If the arrival time at each node $i \in N \cup \{0\}$ belongs to the time interval $[a_i, b_i]$, then the tour and the associated arrival time vector form a **feasible tour**. The problem is to find the feasible tour of minimum cost.*

Consider a feasible tour encoded as:
– an **incidence vector** $x \in \{0,1\}^A$ defined by $x_{ij} = 1$ if arc $(i,j) \in A$ belongs to the tour, $x_{ij} = 0$ otherwise,

– and an **arrival time vector** $w \in \mathbb{Z}_+^{n+1}$, where w_i denotes the arrival time of the tour at node $i \in N \cup \{0\}$.

Question 2 *(5 points).*

Q2.1. linearize the following condition: x *is a path in* G *starting from node* 0*, visiting nodes in* N*, then finishing at node* 0.

Q2.2. linearize the following condition: *each node in* N *appears exactly once in* x

Q2.3. linearize the following condition: $x_{ij} = 1 \implies w_j \geqslant w_i + t_{ij}, \forall (i,j) \in A, i \neq 0$.

Q2.4. show that any tour (x, w) satisfying this last condition **Q2.3** cannot contain any proper subtour

Q2.5. model this problem (TSP-TW) as an Integer Linear Program.

1. at each node i, the number of entering arcs is equal to the number of leaving arcs:
$$\sum_{j|(i,j)\in A} x_{ij} = \sum_{j|(i,j)\in A} x_{ji}, \forall i \in N \cup \{0\}.$$
Furthermore, for $i = 1$:
$$\sum_{j|(0,j)\in A} x_{0j} = 1.$$
2. this number of arcs is equal to 1 for each node i:
$$\sum_{j|(i,j)\in A} x_{ij} = 1, \forall i \in N \cup \{0\}$$
3. for each feasible tour: $w_j - w_i \geqslant l_{ij} = a_j - b_i$ then the condition can be linearized as:
$$w_j - w_i \geqslant (t_{ij} - l_{ij})x_{ij} + l_{ij}, \quad \forall (i,j) \in A, i \neq 0.$$
For $i = 0$, the condition becomes
$$w_j \geqslant (t_{0j} - a_j)x_{0j} + a_j, \quad \forall j \in N,$$
4. Let (x, w) satisfying this last set of constraints and assume that x contains subcycles. Consider a subcycle $(i_1, i_2, \ldots, i_k, i_{k+1} = i_1)$, $k \geqslant 2$ which does not contain the depot 0, i.e. $i_j \in N$, $\forall j = 1, \ldots, k$. By assumption, we have that $w_{i_{j+1}} - w_{i_j} \geqslant t_{i_j i_{j+1}}, \forall j = 1, \ldots, k$. Hence $0 = w_{i_{k+1}} - w_{i_1} = \sum_{j=1}^{k+1} t_{i_j i_{j+1}} > 0$ which is absurd.

$$\begin{aligned}
(TSP - TW) : z \; = \; \min & \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t.} & \sum_{j|(i,j)\in A} x_{ij} = 1 & \forall i \in N \cup \{0\}, \\
& \sum_{j|(j,i)\in A} x_{ji} = 1 & \forall i \in N \cup \{0\}, \\
& w_j - w_i \geqslant (t_{ij} - l_{ij})x_{ij} + l_{ij} & \forall (i,j) \in A, i \neq 0, \\
& w_j \geqslant (t_{0j} - a_j)x_{0j} + a_j & \forall j \in N, \\
& a_i \leqslant w_i \leqslant b_i & \forall i \in N \cup \{0\}, \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in A, \\
& w_i \in \mathbb{R}_+ & \forall i \in N \cup \{0\}.
\end{aligned}$$

3 Change-Making Problem

Problem 3 The Change Making Problem (CMP).
A cashier has to assemble a given change $c \in \mathbb{Z}_{*+}$ *using the less number of coins of specified values* $w_j \in \mathbb{Z}_{*+}$, $j = 1, \ldots, n$. *For each value, an unlimited number of coins is available. We assume, without loss of generality, that the coin values are sorted by decreasing values* $w_1 > w_2 > \cdots > w_n$..

Question 3 *(5 points).*

Q3.1. model this problem as an Integer Linear Program (CMP).

Q3.2. show that $\lceil \frac{c}{w_1} \rceil$ is a lower bound of (CMP).

Q3.3. find an optimum solution of the LP-relaxation $(\bar{P})$ of (CMP).

Q3.4. if $\frac{c}{w_1}$ is integer, then find an optimum solution of (CMP); otherwise, show that $x_1 \leqslant \lfloor \frac{c}{w_1} \rfloor$ defines a cutting-plane for (CMP), where x_1 denotes the number of coins of values w_1.

Q3.5. let $(\bar{P}')$ denote the Linear Program obtained by augmenting $(\bar{P})$ with the cutting-plane of the previous question; show that $(\bar{P}')$ is a relaxation of (CMP).

Q3.6. find an optimum solution of $(\bar{P}')$ with $\lfloor \frac{c}{w_1} \rfloor$ coins of values w_1 and a fractional number of coins of value w_2 to determine, and derive an improved lower bound for (CMP).

1. $$(CMP) : z = \min \sum_{j=1}^{n} x_j \quad \text{s.t.} \quad \sum_{j=1}^{n} w_j x_j = c, \quad x_j \in \mathbb{Z}_+ \; \forall j = 1, \ldots, n \,\}.$$
2. Let x be a feasible solution of (CMP) then $\sum_{j=1}^{n} x_j \geqslant \sum_{j=1}^{n} \frac{w_j}{w_1} x_j = \frac{c}{w_1}$ since $\frac{w_j}{w_1} \leqslant 1$ for all $j = 1, \ldots, n$. As $\sum_{j=1}^{n} x_j$ is integer then $\sum_{j=1}^{n} x_j \geqslant \lceil \frac{c}{w_1} \rceil$.
3. $\bar{x} = (\frac{c}{w_1}, 0, \ldots, 0)$ is an optimum solution of $(\bar{P})$ since it is feasible and, as above, any fractional solution of $(\bar{P})$ is of greatest cost $\sum_{j=1}^{n} x_j \geqslant \frac{c}{w_1}$.
4. If $\frac{c}{w_1}$ is integer then $\bar{x}$ is feasible and then optimum for (CMP). Otherwise, $\bar{x}$ does not satisfy constraint $x_1 \leqslant \lfloor \frac{c}{w_1} \rfloor$. Furthermore, this constraint is a valid inequality for (CMP) since for all feasible solution x of (CMP) we have that x_1 is integer and $x_1 = \frac{c}{w_1} - \sum_{j=1}^{n} \frac{w_j}{w_1} x_j \leqslant \frac{c}{w_1}$.
5. The set of feasible solutions of (CMP) is included in the set of feasible solutions of $(\bar{P}')$ according to the previous question, and the objective function is the same in the two problems.
6. Let $\bar{c} = c - \lfloor \frac{c}{w_1} \rfloor w_1$, then $\bar{x}' = (\lfloor \frac{c}{w_1} \rfloor, \frac{\bar{c}}{w_2}, 0, \ldots, 0)$ is feasible for $(\bar{P}')$ and optimum since, for all feasible solution x of $(\bar{P}')$:
$$\begin{aligned}
\sum_{j=1}^{n} x_j \geqslant x_1 + \sum_{j=2}^{n} \frac{w_j}{w_2} x_j = x_1 + \frac{c - w_1 x_1}{w_2} = \frac{c}{w_2} - \frac{w_1 - w_2}{w_2} x_1 \\
\geqslant \frac{c}{w_2} - \frac{w_1 - w_2}{w_2} \lfloor \frac{c}{w_1} \rfloor = \lfloor \frac{c}{w_1} \rfloor + \frac{\bar{c}}{w_2} = \sum_{j=1}^{n} \bar{x}'_j.
\end{aligned}$$
As a consequence $\lfloor \frac{c}{w_1} \rfloor + \lceil \frac{\bar{c}}{w_2} \rceil$ is a lower bound for (CMP) which is as least as good as the previous bound $\lceil \frac{c}{w_1} \rceil$ since $\bar{c} > 0$ and then:
$$\lfloor \frac{c}{w_1} \rfloor + \lceil \frac{\bar{c}}{w_2} \rceil = \lceil \frac{c}{w_1} \rceil + \lceil \frac{\bar{c}}{w_2} \rceil - 1 \geqslant \lceil \frac{c}{w_1} \rceil.$$

4 Capacitated Facility Location Problem (CFL)

Problem 4 The Capacitated Facility Location Problem (CFL).
Given a set of potential depots $J = \{1, \dots, n\}$ and a set of clients $I = \{1, \dots, m\}$, there is a number of items, all of the same type, to serve from the depots to the clients. Each depot $j \in J$ has a finite capacity b_j (the number of items available at the depot) and each client $i \in I$ has a finite requirement a_i (the minimum number of items ordered by the client). There is a fixed cost f_j associated with the use of depot $j \in J$ and an unit transportation cost c_{ij} that is paid for each item served to client $i \in I$ from depot $j \in J$. All data are positive integers. The problem is to decide which depots to open and which amount of items to serve to each client from each open depot, so as to minimize the sum of the fixed and transportation costs.

Question 4 ***(5 points).***

Q4.1. model this problem as an Integer Linear Program (CFL);

Q4.2. formulate the lagrangian relaxation of (CFL) when dualizing the client requirement constraints;

Q4.3. show that each lagrangian subproblem can be decomposed into independent subproblems;

Q4.4. assume that the requirements for all clients are equal to 1; propose an algorithm to solve or to approximate this lagrangian relaxation.

1.

$$
\begin{aligned}
(CFL): z \;=\; \min & \sum_{i \in I}\sum_{j \in J} c_{ij}x_{ij} + \sum_{j \in J} f_j y_j \\
\text{s.t.} & \sum_{j \in J} x_{ij} \geqslant a_i && \forall i \in I, \\
& \sum_{i \in I} x_{ij} \leqslant b_j y_j && \forall j \in J, \\
& x_{ij} \in \mathbb{Z}_+ && \forall i \in I, \forall j \in J, \\
& y_j \in \{0,1\} && \forall j \in J.
\end{aligned}
$$

2. For each multiplier $\lambda \in \mathbb{R}_+^I$:

$$
\begin{aligned}
(L_\lambda): z_\lambda \;=\; \min & \sum_{i \in I}\sum_{j \in J} (c_{ij} - \lambda_i)x_{ij} + \sum_{j \in J} f_j y_j + \sum_{i \in I} \lambda_i a_i \\
\text{s.t.} & \sum_{i \in I} x_{ij} \leqslant b_j y_j && \forall j \in J, \\
& x_{ij} \in \mathbb{Z}_+ && \forall i \in I, \forall j \in J, \\
& y_j \in \{0,1\} && \forall j \in J.
\end{aligned}
$$

The lagrangian dual problem is $D = \max\{z_\lambda \mid \lambda \in \mathbb{R}_+^I\}$.

3. (L_λ) can be decomposed into n problems, one for each depot: $z_\lambda = \sum_{i \in I} \lambda_i a_i + \sum_{j \in J} u_\lambda^j$ where u_λ^j is defined for all $j \in J$ by:

$$
\begin{aligned}
(L_\lambda^j): u_\lambda^j \;=\; \min & \sum_{i \in I} (c_{ij} - \lambda_i)x_{ij} + f_j y_j \\
\text{s.t.} & \sum_{i \in I} x_{ij} \leqslant b_j y_j \\
& x_{ij} \in \mathbb{Z}_+ && \forall i \in I, \\
& y_j \in \{0,1\}.
\end{aligned}
$$

4. Consider a subproblem (L_λ^j). If depot j is not open then $y_j = 0$ and $x_{ij} = 0\ \forall i \in I$ is the unique feasible solution and its cost is 0. Conversely, if depot j is open then it should serve each profitable client in the limit of its capacity: the subproblem is an Integer Knapsack Problem. When the requirements are all equal to 1 then the subproblem becomes a $0-1$ Knapsack Problem where all weights are equal to 1. An optimum solution consists in sorting the clients by increasing order of cost $c_{ij} - \lambda_i$ then selecting the first at most b_j clients having negative costs $c_{ij} - \lambda_i < 0$. Let $I_\lambda^j \subseteq I$ be the set of selected clients. The optimum value of (L_λ^j) is then $u_\lambda^j = \min\{0, \sum_{i \in I_\lambda^j}(c_{ij} - \lambda_i) + f_j\}$ and it can be computed in $O(m)$ time. As a consequence, each subproblem (L_λ) can be solved at optimality and its optimum value $z_\lambda = \sum_{i \in I} \lambda_i a_i + \sum_{j \in J} u_\lambda^j$ can be computed in $O(mn)$ time. Solving the dual lagrangian problem requires a priori to solve iteratively subproblems (L_λ) for different values of $\lambda \in \mathbb{R}$. The sequence of multipliers can be chosen according to a cutting-plane algorithm or a subgradient algorithm.