M2 ORO: Advanced Integer Programming
Final Exam – 2nd session

january 18, 2010

duration: 2 hours.

documents: lecture notes are authorized. No book, no book copy.

grades: 20 points = 3 points (Modeling) + 17 points (Relaxations).
The bonus questions tagged with $(*)$ are optional.

Notations:

$conv(X)$	the convex hull of X
$\mathbb{R}, \mathbb{R}_+, \mathbb{R}_{*+}$	the sets of real, non-negative real, and positive real numbers

---

---

1 Modeling

Problem 1 *Given a certain number of regions, the problem is to decide where to install a set of emergency service centers. For each possible location, we know the cost of installing a service center at this location, and which regions (in the neighborhood) it will be able to service. The goal is to choose a minimum cost set of service centers so that each region is covered.*

Question 1 ***(3 points).***

Q1.1. Formulate this problem as an Integer Linear Program.

2 Relaxations

Problem 2 The Generalized Assignment Problem (GAP).

Assign n items to m knapsacks, with:

$p_{ij} =$ *profit of item j if assigned to knapsack i,*

$w_{ij} =$ *weight of item j if assigned to knapsack i,*

$c_i =$ *capacity of knapsack i,*

such that:

(1) *each item is assigned to exactly one knapsack,*

(2) *the total weight assigned to a knapsack does not exceed its capacity,*

(3) *the sum of the profits is maximized.*

Question 2 ***(17 points).***

Q2.1. Model problem (GAP) as an Integer Linear Program.

2.1 Relaxation of the semi-assignment constraints

Consider the variant (LEGAP) of problem (GAP) where condition (1) is replaced by:

(1′) *each item is assigned to at most one knapsack.*

Q2.2. Compare the feasibility of (GAP) and (LEGAP).

Q2.3. Name the problem to which (LEGAP) reduces when the number of knapsacks is $m = 1$; name the problem to which (LEGAP) reduces when the cost and profit of each item are independent of the knapsack it is assigned to.

Q2.4.(∗) Show that any instance (n, m, p, w, c) of (LEGAP) is equivalent to an instance (n', m', p', w', c') of (GAP) such that $n' = n$ and $m' = m + 1$ (define p', w', and c').

2.2 Relaxation of the capacity constraints

Consider (R_0) the combinatorial relaxation of (GAP) obtained by removing the capacity constraints (2), and let $\bar{N}$ be an optimum solution of (R_0), defined as follows: for each knapsack i, $\bar{N}_i$ denotes the set of items assigned to i. Furthermore, $\bar{M}$ denotes the set of knapsacks whose capacity is exceeded in the relaxed solution $\bar{N}$.

Q2.5. Express the optimum value u_0 of (R_0).

Q2.6. Express a condition under which, $\bar{N}$ is a feasible solution of (GAP).

Note that to make $\bar{N}$ feasible for (GAP), a number of items assigned to knapsacks with violated capacity have to be reassigned. This observation will allow us to compute an improved upper bound u_1 on u_0.

Q2.7. For each knapsack $i \in \bar{M}$ with violated capacity and for each item $j \in \bar{N}_i$ assigned to i, consider a feasible solution $\bar{N}(j)$ of (R_0) obtained from $\bar{N}$ by reassigning item j to any knapsack other than i. Estimate the maximum value $u(j)$ of solution $\bar{N}(j)$ compared to the value u_0 of $\bar{N}$.
Let $q_j = u_0 - u(j)$ denote the minimum penalty that is incurred to the profit of $\bar{N}$ if item j is reassigned.

Q2.8. For $i \in \bar{M}$, consider the following 0-1 Knapsack Problem in minimization form:

$$\begin{aligned}(KP_i): v_i = \min & \sum_{j \in \bar{N}_i} q_j y_j \\ \text{s.t.} & \sum_{j \in \bar{N}_i} w_{ij} y_j \geqslant \Big(\sum_{j \in \bar{N}_i} w_{ij} - c_i\Big) \\ & y_j \in \{0,1\} \qquad \forall j \in \bar{N}_i.\end{aligned}$$

Interpret the value v_i in relation to $\bar{N}$. (*hint: exhibit a solution of (R_0) obtained from $\bar{N}$ and an optimum solution of (KP_i).*)

Q2.9.(∗) Show that $u_1 = u_0 - \sum_{i \in \bar{M}} v_i$ is an upper bound for (GAP).

Q2.10. For $i \in \bar{M}$, transform (KP_i) into an equivalent 0-1 Knapsack Problem in maximization form. (*hint: transform each binary variable y_j into a new binary variable z_j.*)

Q2.11. Compute u_0 and u_1 for the instance of (GAP) defined by: $n = 7$, $m = 2$,

$$p = \begin{pmatrix} 6 & 9 & 4 & 2 & 10 & 3 & 6 \\ 4 & 8 & 9 & 1 & 7 & 5 & 4 \end{pmatrix},\ w = \begin{pmatrix} 4 & 1 & 2 & 1 & 4 & 3 & 8 \\ 9 & 9 & 8 & 1 & 3 & 8 & 7 \end{pmatrix},\ c = \begin{pmatrix} 11 \\ 22 \end{pmatrix}.$$

2.3 Dualization of the capacity constraints

Consider the lagrangian relaxation of (GAP) obtained by dualizing the capacity constraints (2).

Q2.12. Formulate the dual lagrangian relaxation in the following form:

$$(R_2): u_2 = \max\{fx \mid Ex \leqslant d,\ x \in conv(X)\},$$

where $Ex \leqslant d$ are the dualized constraints, and $X \in \{0,1\}^{m \times n}$ is the set of feasible solutions of (R_0).

Q2.13. Let $\bar{X} \in \mathbb{R}^{m \times n}$ be the set of solutions of the LP relaxation of (R_0); show that $X = \bar{X}$.

Q2.14. Show that u_2 is equal to the optimum value $\bar{z}$ of the LP relaxation of (GAP).

Q2.15. Express the optimum value of a lagrangian subproblem for any given multipliers.

2.4 Dualization of the assignment constraints

Consider the lagrangian relaxation of (GAP) obtained by dualizing the assignment constraints (1).

Q2.16. Show that each lagrangian subproblem can be decomposed into independent subproblems (name these subproblems).

Q2.17. Consider the optimum u^{λ} of the lagrangian subproblem associated to the multipliers λ defined by:

$$\lambda_j = \max\{p_{ij} \mid i = 1, \ldots, m\} - q_j, \forall j = 1, \ldots, n$$

where q_j is defined in question **Q2.7**. According to **Q2.10** and **Q2.16**, show that $u^{\lambda} = u_1$.

Q2.18. Compare the bounds u_0, u_1, u_2 in terms of quality and complexity.