M2 ORO: Advanced Integer Programming
Solution Final Exam – 2nd session

january 18, 2011

duration: 1h00.

documents: lecture notes are authorized. No book, no book copy.

grades: 3 problems of respectively 3+3+4 points each = 10 points.

Notations:

$\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_{+*}$	the sets of integer, non-negative integer, and positive integer numbers
$\mathbb{R}, \mathbb{R}_+, \mathbb{R}_{+*}$	the sets of real, non-negative real, and positive real numbers

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1 Model of graph

Problem 1 The Graph Bandwidth Problem.
Consider an undirected connected graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges. A linear layout *of G is a numbering of the vertices of G. In other words, it is an assignment $f : V \to \{1, 2, \dots, n\}$, such that different vertices $u, v \in V$ have different numbers $f(u) \neq f(v)$. The* bandwidth *of layout f, denoted by $\Phi_f(G)$, is the maximum difference between the numbers assigned to adjacent vertices, i.e. $\Phi_f(G) = \max\{|f(u) - f(v)|, (u, v) \in E\}$. The Graph Bandwidth Problem is to find the minimum bandwidth over all possible linear layouts of G. This value, denoted by $\Phi(G)$, is called the bandwidth of G.*

Question 1 ***(3 points).***

Q1.1. Model this problem as an Integer Linear Program.

Q1.2. Compute the optimal value of its continuous relaxation.

1.

$$
\begin{aligned}
(GB):\ & \min z \\
\text{s.t. } & z \geqslant \sum_{k=1}^{n} k(x_u^k - x_v^k) && \forall (u,v) \in E \\
& z \geqslant \sum_{k=1}^{n} k(x_u^k - x_v^k) && \forall (u,v) \in E \\
& \sum_{k=1}^{n} x_u^k = 1 && \forall u \in V, \\
& \sum_{u \in V} x_u^k = 1 && \forall k = 1, \dots, n, \\
& x_u^k \in \{0,1\} && \forall u \in V, k = 1, \dots, n \\
& z \geqslant 0.
\end{aligned}
$$

where, in any solution, z is the bandwith of the layout f defined by $x_u^k = 1 \iff f(u) = k$, or alternatively by $f(u) = \sum_{k=1}^{n} k x_u^k$ for all $u \in V$.

2. The continuous relaxation of (GB) has an optimal value of 0. It is the cost of the optimal solution: $x_u^k = 1/n$ for all $u \in V$, $k = 1, \dots, n$.

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2 Model of logic

Problem 2 *Suppose you are interested in choosing a set of investments among seven possible investments numbered from 1 to 7. The estimate profit of each investment $i = 1..7$ is given by a positive integer $c_i \in \mathbb{Z}_{*+}$. You want to maximize your profit, knowing that:*

1. *you cannot invest in all of them*
2. *you must choose at least one of them*
3. *at most one of investments 1 and 3 can be chosen*
4. *investment 4 can be chosen only if investment 2 is also chosen*
5. *you must choose either both of investments 1 and 5, or neither*
6. *you must choose at least one of investments 1,2,3 or at least two of investments 2,4,5,6*

Question 2 ***(3 points).***

Q2.1. Model this problem as a Binary Integer Linear Program, including all the individual constraints above.

Q2.2. Simplify your model by fixing variables and removing redundant inequalities.

1.

$$
\begin{aligned}
\max \sum_{i=1}^{7} c_i x_i & & \\
\text{s.t.} \sum_{i=1}^{7} x_i \leqslant 6 & & (1)\\
\sum_{i=1}^{7} x_i \geqslant 1 & & (2)\\
x_1 + x_3 \leqslant 1 & & (3)\\
x_4 \leqslant x_2 & & (4)\\
x_1 = x_5 & & (5)\\
x_1 + x_2 + x_3 \leqslant 3y_1 & & (6a)\\
y_1 \leqslant x_1 + x_2 + x_3 & & (6b)\\
x_2 + x_4 + x_5 + x_6 \leqslant 4y_2 & & (6c)\\
2y_2 \leqslant x_2 + x_4 + x_5 + x_6 & & (6d)\\
y_1 + y_2 \geqslant 1 & & (6e)\\
x_i \in \{0,1\} & & i = 1..7\\
y_i \in \{0,1\} & & i = 1..2
\end{aligned}
$$

where $x_i = 1$ iff investment i is chosen, $y_1 = 1$ iff at least one of investments 1,2,3 is chosen, and $y_2 = 1$ iff at least two of investments 2,4,5,6 are chosen.

2. equality (5) can be removed by merging the two variable $x_1 = x_5$ and x_7 can be fixed to 1 in any optimal solution (since x_7 is unconstrained and $c_7 > 0$).

3. inequalities (1) and (2) can be removed: (1) is implied by (3), (2) is implied by (6). Note that (2) is also implied by the objective, because: this problem has feasible non-zero solutions, e.g. $(1, 0, 0, 0, 1, 0)$, with positive cost, then all optimal solutions are non-zero and satisfy (2).

4. to model constraint (6), one can relax the definitions of the y variables as: $y_1 = 1$ if at least one of investments 1,2,3 is chosen, and $y_2 = 1$ if at least two of investments 2,4,5,6 are chosen. As a consequence, we remove inequalities (6a) and (6c). Actually, (6e) enforces either y_1 or y_2 to be equal to 1, i.e. either $x_1 + x_2 + x_3 \geqslant 1$ or $x_2 + x_4 + x_5 + x_6 \geqslant 2$.

5.

$$
\begin{aligned}
c_7 + \max\ (c_1 + c_5)x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_6x_6 & & \\
\text{s.t. } x_1 + x_3 \leqslant 1 & & (3)\\
x_4 \leqslant x_2 & & (4)\\
y_1 \leqslant x_1 + x_2 + x_3 & & (6b)\\
2y_2 \leqslant x_1 + x_2 + x_4 + x_6 & & (6d)\\
y_1 + y_2 \geqslant 1 & & (6e)\\
x_i \in \{0,1\} & & i = 1,2,3,4,6\\
y_i \in \{0,1\} & & i = 1,2
\end{aligned}
$$

3 Lagrangian Relaxation

Problem 3 Resource Constrained Shortest Path Problem (RCSPP).

Let $G(V, A)$ be a directed acyclic graph with a source $s \in V$ and a sink $t \in V$, and let R be a set of resources with limited capacities $C_r \in \mathbb{Z}_{+*}$, $\forall r \in R$. Each arc $a \in A$ has a distance $d_a \in \mathbb{Z}_+$, and traversing arc a consumes a given amount $c_{ar} \in \mathbb{Z}_+$ of each resource $r \in R$. The RCSPP is to find a path from s to t of minimal distance and such that, for each resource $r \in R$, the total amount of resource r consumed along the path does not exceed the capacity C_r.

Question 3 ***(4 points).***

Q3.1. Model this problem as a Binary Integer Linear Program using decision variables $x_a = 1$ if arc a belongs to the selected path, and $x_a = 0$ otherwise.

Q3.2. Describe a lagrangian relaxation applied to this model by dualizing the resource constraints and exhibit the nature of the sub-problems and their complexity.

Q3.3. Compare the optimum of the lagrangian dual with the optimum of the LP-relaxation.

$$
\begin{aligned}
(P)\ z = \min \sum_{a \in A} d_a x_a & & \\
\sum_{a \in \delta_-(i)} x_a - \sum_{a \in \delta_+(i)} x_a = b_i & & i \in V\\
\sum_{a \in A} c_{ar} x_a \leqslant C_r & & r \in R\\
x_a \in \{0,1\} & & a \in A
\end{aligned}
$$

where $b_i = 1$ if $i = s$, $b_i = -1$ if $i = t$, $b_i = 0$ otherwise. Let dualize the resource constraints, then the lagrangian dual is $(D) : \max_{\lambda \in \mathbb{R}_+^R} (z_\lambda - \sum_{r \in R} C_r \lambda_r)$ where:

$$
\begin{aligned}
(P_\lambda)\ z_\lambda = \min \sum_{a \in A} (d_a + \sum_{r \in R} \lambda_r c_{ar}) x_a & & \\
\sum_{a \in \delta_-(i)} x_a - \sum_{a \in \delta_+(i)} x_a = b_i & & i \in V\\
x_a \in \{0,1\} & & a \in A
\end{aligned}
$$

Such sub-problem is polynomially solvable as it is a shortest path problem from s to t on G using a distance value $d_a + \sum_{r \in R} \lambda_r c_{ar}$ on each arc $a \in A$. Actually, model (P_λ) is ideal (its coefficient matrix is a flow matrix) hence the optimum of (D) is the optimum of the LP-relaxation.