M2 ORO: Advanced Integer Programming
Solution Final Exam – 1st session

november 21, 2011

duration: 1h30.

documents: lecture notes are authorized. No book, no book copy.

grades: 2 problems of respectively 10 points each.

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1 Modeling and decompositions

Problem 1 Multicommodity Flow Problems.
Consider a directed graph $G = (V, A)$ representing a communication network and a set K of messages to be routed in this network. Each arc $(i, j) \in A$ of the graph has an associated routing cost $c_{ij} \geqslant 0$ and an upper capacity $u_{ij} \geqslant 0$. Each message $k \in K$ is defined by an origin-destination node pair $(s_k, t_k) \in V \times V$ and a size $d_k \geqslant 0$. Routing a message $k \in K$ in G is to send exactly d_k units of flow from the origin node s_k to the destination node t_k through the network. Each unit of flow is unsplittable and induces a cost c_{ij} on each arc $(i, j) \in A$ it travels.

The Minimum-Cost Multicommodity Flow Problem (MCMF) is to route all the messages at a minimum cost such that the total amount of flows travelling on a given arc does not exceed its capacity.

Question 1 ***(10 points).***

Q1.1. Model the MCMF problem as an Integer Linear Program.

Q1.2. Choose a set of constraints to dualize and formulate the corresponding lagrangian dual; justify your choice.

Q1.3. Show how the relaxed constraints could be partly re-enforced in the lagrangian subproblems without changing the complexity of these problems.

Now, suppose that each message $k \in K$ is unsplittable, that is: the d_k units of flow must travel through an unique route from s_k to t_k. This variant is called the Minimum-Cost Unsplittable Multicommodity Flow Problem (MCUMF).

Q1.4. Modify the MCMF Integer Linear Program so as to obtain a model of the MCUMF as a Binary Linear Program.

Q1.5. Considering the same dualization, identify the problem class of the lagrangian subproblems and a well-known algorithm to solve them.

Finally, suppose that each message $k \in K$ is provided with a restricted set P_k of possible paths from s_k to t_k, the problem is now to select exactly one path in P_k to route each message k at a minimum cost.

Q1.6. Model this latter variant as a Binary Linear Program.

Q1.7. Deduce from this model a column-generation approach for the MCMF.

1. Let $X_{(f,s,t)} = \left\{x \in Z_+^A \mid \sum_{a \in A^+(i)} x_a - \sum_{a \in A^-(i)} x_a = \begin{cases} f & \text{if } i = s, \\ -f & \text{if } i = t, \\ 0 & \text{otherwise.} \end{cases} \right\}$

   $X_{(f,s,t)}$ models the set of feasible flows of value f from s to t in G. Let x_a^k be the amount of flow of message $k \in K$ on arc $a \in A$, then:

$$
\begin{aligned}
(MCMF): z = \min \sum_{k \in K} \sum_{a \in A} c_a x_a^k && && (1) \\
\text{s.t.} \sum_{k \in K} x_a^k \leqslant u_a && \forall a \in A, && (2) \\
x^k \in X_{(d_k, s_k, t_k)} && \forall k \in K. && (3)
\end{aligned}
$$

2. the capacity constraints (2) are coupling constraints as their dualization allows to split the lagrangian subproblems in several easy problems: one min-cost flow problem for each message $k \in K$. Formally, the lagrangian dual corresponding to this dualization is $(L): \max\{\sum_{k \in K} z_\mu^k - \sum_{a \in A} \mu_a u_a \mid \mu \in \mathbb{R}_+^A\}$ where $z_\mu^k = \min_{x \in X_{(d_k, s_k, t_k)}} \sum_{a \in A} (c_a + \mu_a) x_a$ is the minimum cost flow of value d_k from s_k to t_k on the weighted graph $G_\mu = (V, A, c + \mu)$ and without capacity.

3. Consider the addition of the redundant constraints $\{x_a^k \leqslant u_a, \forall k \in K, \forall a \in A\}$ in the $(MCMF)$ model. Then the lagrangian subproblems are modified by considering an upper capacity u_a on each arc of the flow network G'. The problem of finding a flow of minimum cost remains polynomial.

4. let $y_a^k = 1$ if message k is routed on arc a, and $y_a^k = 0$ otherwise, then:

$$
\begin{aligned}
(MCUMF): z = \min \sum_{k \in K} \sum_{a \in A} c_a d_k y_a^k && \\
\text{s.t.} \sum_{k \in K} d_k y_a^k \leqslant u_a && \forall a \in A, \\
y^k \in X_{(1, s_k, t_k)} && \forall k \in K.
\end{aligned}
$$

5. the lagrangian subproblems are to compute, for each $k \in K$, a shortest path from s_k to t_k in G_μ.

6. The last variant of the MCUMF can be modeled as a constrained set covering problem with $z_p^k = 1$ if message $k \in K$ is routed on path $P \in P_k$ and $z_p^k = 0$ otherwise:

$$
\begin{aligned}
(MCUMF'): z = \min \sum_{k \in K} c_p d_k z_p^k && \\
\text{s.t.} \sum_{k \in K} \sum_{p \in P_k} d_k \delta_a^p z_p^k \leqslant u_a && \forall a \in A, \\
\sum_{p \in P_k} z_p^k \geqslant 1 && \forall k \in K \\
z_p^k \in \{0, 1\} && \forall k \in K, p \in P_k,
\end{aligned}
$$

   where for any path p in G, δ_a^p are the set indicators $\delta_a^p = 1$ if arc $a \in A$ belongs to p and $\delta_a^p = 1$ otherwise, and $c_p = \sum_{a \in p} c_a$ its cost. Note that $\sum_{p \in P_k} z_p^k = 1$ is satisfied at optimality.

7. The MCMF can be modeled the same way by considering for each $k \in K$, P_k the set of all paths

from s_k to t_k in G:

$$(\text{MCMF}'): z = \min \sum_{k\in K} c_p \xi_p^k \quad (4)$$

$$\text{s.t.} \sum_{k\in K}\sum_{p\in P_k} \delta_a^p \xi_p^k \leqslant u_a \quad \forall a \in A, \quad (5)$$

$$\sum_{p\in P_k} \xi_p^k \geqslant d_k \quad \forall k \in K, \quad (6)$$

$$\xi_p^k \in \mathbb{Z}_+ \quad \forall k \in K, p \in P_k. \quad (7)$$

Each set P_k can be generated progressively, in a column generation fashion: starting from an arbitrary set of paths from s_k to t_k. At each iteration, and for each $k \in K$ the subproblem to solve is to find a path p from s_k to t_k in graph $G^\lambda = (V, A, c+\lambda)$ with a cost strictly lower than μ_k, where λ and μ are the dual values in the master program of constraints (5) and (6), respectively. The subproblem consists in computing a shortest path and checking its value against μ_k.

2 Cutting-planes

Problem 2 Prize Collecting Traveling Salesman Problem (PCTSP).
Consider a variant of the TSP on an undirected graph $G = (V, E)$ in which the salesman:

- *makes a profit $f_j \geqslant 0$ if he visits city (or node) $j \in V$,*
- *pays a travel cost c_e if he traverses edge $e \in E$,*
- *starts and ends his tour at city $0 \in V$, visits at least two other cities, but is **not obliged to visit all the cities**.*

The Prize Collecting Traveling Salesman Problem is to find a tour that maximizes the residual profit (the sum of the profits minus the sum of the costs). We introduce the following notations:

- *$E(i)$ is the set of edges incident to node $i \in V$*
- *$E(S)$ is the set of edges with their two extremities in the subset of nodes $S \subseteq V$*
- $V' = V \setminus \{0\}$
- $E' = E \setminus E(0)$

Question 2 ***(10 points).***

Q2.1. Model this problem as a Binary Linear Program.

Consider a model (P) on two vectors of binary variables $x \in \{0,1\}^E$ and $y \in \{0,1\}^V$, defined by $x_e = 1$ if and only if edge $e \in E$ belongs to the salesman tour and $y_i = 1$ if and only if city i is visited by the salesman. The model includes the following set of constraints:

$$\sum_{e\in E(S)} x_e \leqslant \sum_{i\in S\setminus\{k\}} y_i, \quad \forall S \subseteq V', \forall k \in S. \quad (8)$$

Q2.2. Show that these constraints are valid and explain their role (8).

We want to solve the linear relaxation of model (P). As there is an exponential number of constraints (8), we want to generate them on the fly, using a cutting-plane algorithm: at each iteration, only a restricted number of constraints (8) are present in the model and a solution (x^*, y^*) of the LP-relaxation is computed. Then the separation problem is solved in order to find one new constraint (8) violated by (x^*, y^*). It can be shown that the separation problem is equivalent to find $k \in V'$ such that $\zeta_k > 0$ with:

$$(S_k): \quad \zeta_k = \max\Big\{ \sum_{e=(i,j)\in E', i<j} x_e^* z_i z_j - \sum_{i\in V'\setminus\{k\}} y_i^* z_i \ \Big|\ z \in \{0,1\}^{V'}, z_k = 1 \Big\}. \quad (9)$$

Q2.3. Show that the Binary Quadratic Program (S_k) can be modeled as a Binary Linear Program (L_k) (*hint: introduce a binary variable defined by $w_e = z_i z_j$ for each edge $e = (i,j) \in E'$, $i < j$*).

Q2.4. Show that the constraints enforcing $z_i = 1, z_j = 1 \Rightarrow w_e = 1 \ \forall e = (i,j) \in E'$ in model (L_k) are redundant.

Q2.5. Deduce the class of complexity of the separation problem.

$$(P): \max \sum_{i\in V} f_i y_i - \sum_{e\in E} c_e x_e$$

$$\text{s.t.} \sum_{e\in E(i)} x_e = 2y_i \quad \forall i \in V$$

$$\sum_{e\in E(S)} x_e \leqslant \sum_{i\in S\setminus\{k\}} y_i \quad \forall S \subseteq V', \forall k \in S$$

$$y_i \in \{0,1\}, y_0 = 1 \quad \forall i \in V'$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

1. Consider $T \subseteq E$ a feasible tour and (x, y) its encoding in the ILP. For any non-empty subset of nodes $S \subseteq V'$, $T \cap E(S)$ is a set of $n \geqslant 0$ disjoint paths, each path p having $m_p \geqslant 1$ vertices in S for $m_p - 1$ edges. If $n = 0$ then $x_e = 0$ for all $e \in E(S)$ and $y_i = 0$ for all $i \in S$ and constraint (8) is satisfied for any $k \in S$. If $n \geqslant 1$, then $\sum_{e\in E(S)} x_e = \sum_{p=1}^n (m_p - 1) = \sum_{p=1}^n m_p - n \leqslant \sum_{p=1}^n m_p - 1 = \sum_{i\in S} y_i - 1 \leqslant \sum_{i\in S\setminus\{k\}} y_i$. Then constraints (8) are valid for the problem. These constraints ensure that there is no subtour in V'. Indeed, if a (non-feasible) solution (x, y) has a circuit on the set of nodes $T \subseteq V'$ then $\sum_{e\in E(T)} x_e = \sum_{i\in T} y_i > \sum_{i\in T\setminus\{k\}} y_i$ for all $k \in T$.
2. Assume that (x^*, y^*) violates a subtour constraint for some $S \subseteq V'$, $k \in S$, let $z_i = 1$ iff $i \in S$, then: $\zeta_k \geqslant \sum_{e=(i,j)\in E', i<j} x_e^* z_i z_j - \sum_{i\in V'\setminus\{k\}} y_i^* z_i = \sum_{e\in E(S)} x_e^* - \sum_{i\in S\setminus\{k\}} y_i^* > 0$. Conversely, note that (S_k) is bounded then there exists an optimal solution z^k of value ζ_k for all $k \in V'$. If $\zeta_k > 0$ for some $k \in V'$, let $S = \{i \in V' \mid z_i^k = 1\}$, then $k \in S$, $0 < \zeta_k = \sum_{e\in E(S)} x_e^* - \sum_{i\in S\setminus\{k\}} y_i^* \leqslant \sum_{e\in E(S\cup\{k\})} x_e^* - \sum_{i\in S\setminus\{k\}} y_i^*$ and constraint (S, k) is violated.
3. (S_K) is equivalent to the following program:

$$(L_k): \zeta_k = \max \sum_{e\in E'} x_e^* w_e - \sum_{i\in V'\setminus\{k\}} y_i^* z_i \quad (10)$$

$$\text{s.t. } w_e \leqslant z_i \quad \forall e \in E', \forall i \in e \quad (11)$$

$$w_e \geqslant z_i + z_j - 1 \quad \forall e = (i,j) \in E', i < j \quad (12)$$

$$z_i \in \{0,1\}, z_k = 1 \quad \forall i \in V' \quad (13)$$

$$w_e \in \{0,1\} \quad \forall e \in E' \quad (14)$$

4. Let (L_k') be the program (L_k) after dropping constraints (12) and $\zeta_k' \geqslant \zeta_k$ its optimum. Let (w, z) be an optimal solution of (L_k') such that $z_i = 1$, $z_j = 1$ and $w_e = 0$ for some $e = (i,j) \in E'$, then increasing w_e by 1 leads to a feasible solution of (L_k) with cost $\zeta_k' + x_e^* \geqslant \zeta_k'$. As a consequence $\zeta_k' = \zeta_k$. Note that constraints $z_k \geqslant 1$ and $w_e \leqslant 1$ can also be relaxed.
5. Each row $e \in E'$ of (L_k') contains two non-zero coefficients: 1 in some column $e \in E'$ and -1 in some column $i \in V'$. All other rows contain only one non-zero coefficient 1. The coefficient matrix (and its transpose) is totally unimodular and the right hand side are integer, then formulation (L_k') is ideal. Its LP-relaxation, which can be solved in polynomial time, has only integer optimum solutions.