Centre de Mathématiques Appliquées

Modelling in Linear Programming

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1 Exercises

1.1 Nuclear waste management

A company eliminates nuclear wastes of 2 types A and B, by applying a sequence of 3 processes I, II and III in any order. The processes I, II, III, have limited availability, respectively: 450h, 350h, and 200h per month. The unit processing times depend on the process and waste type, as reported in the following table:

process	I	II	III
waste A	1h	2h	1h
waste B	3h	1h	1h

(first entry reads *one unit of A-type waste is processed in 1 hour with process I*) The profit for the company is 4000 euros to eliminate one unit of waste A and 8000 euros to eliminate one unit of waste B.

Objective: maximize the profit.

$$\begin{aligned} \max\ & 4x_A + 8x_B \\ \text{s.t.}\ & x_A + 3x_B \le 450 \\ & 2x_A + x_B \le 350 \\ & x_A + x_B \le 200 \\ & x_A, x_B \ge 0 \end{aligned}$$

1.2 The two crude petroleum problem [Ralphs]

A petroleum company distills crude imported from Kuwait (9000 barrels available at 20€ each) and from Venezuela (6000 barrels available at 15€ each), to produce gasoline (2000 barrels), jet fuel (1500 barrels), and lubricant (500 barrels) in the following proportions:

	gasoline	jet fuel	lubricant
Kuwait	0.3	0.4	0.2
Venezuela	0.4	0.2	0.3

(first entry reads: *producing 1 unit of gasoline requires 0.3 units of crude from Kuwait*)

Objective: minimize the production cost.

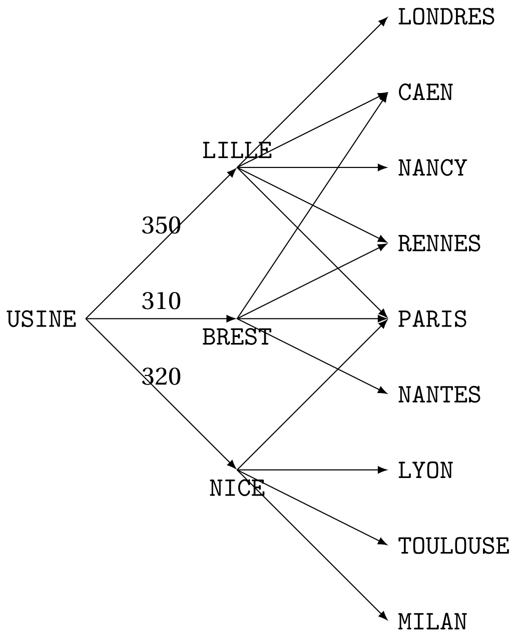
$$\begin{aligned} \min\ & 20x_K + 15x_V \\ \text{s.t.}\ & 0.3x_K + 0.4x_V \ge 2 \\ & 0.4x_K + 0.2x_V \ge 1.5 \\ & 0.2x_K + 0.3x_V \ge 0.5 \\ & 0 \le x_K \le 9 \\ & 0 \le x_V \le 6 \end{aligned}$$

1.3 network flow

A company delivers retail stores in 9 cities in Europe from its unique factory *USINE*. How to manage production and transportation in order to:

- meet the demand of each store,
- not exceed the production limit,
- not exceed the line capacities,
- minimize the transportation costs ?

```
demand = {
    'PARIS': 110,
    'CAEN': 90,
    'RENNES': 60,
    'NANCY': 90,
    'LYON': 80,
    'TOULOUSE': 50,
    'NANTES': 50,
    'LONDRES': 70,
    'MILAN': 70
}
LINES, unitary_cost, capacity = multidict({
    ('USINE','LILLE'): [2.9, 350],
    ('USINE','NICE') : [3.5, 320],
    ('USINE','BREST') : [3.1, 310],
    ('LILLE', 'PARIS') : [1.1, 150],
    ('LILLE','CAEN'): [0.7, 150],
    ('LILLE','RENNES'): [1.0, 150],
    ('LILLE','NANCY'): [1.3, 150],
    ('LILLE','LONDRES'): [1.3, 150],
    ('NICE','LYON'): [0.8, 200],
    ('NICE','TOULOUSE'): [0.2, 110],
    ('NICE','PARIS'): [1.3, 100],
    ('NICE','MILAN'): [1.3, 150],
    ('BREST','NANTES'): [0.9, 150],
    ('BREST','CAEN'):[0.8, 200],
    ('BREST','RENNES'): [0.8, 150],
    ('BREST','PARIS'): [0.9, 100]
})
MAX_PRODUCTION = 900
```



- x_ℓ the quantity of products (*flow*) transported on line $\ell = (i,j) \in \texttt{LINES}$
- $\texttt{TRANSITS} = \{\texttt{LILLE}, \texttt{NICE}, \texttt{BREST}\}$

$$
\begin{aligned}
\min \quad & \sum_{\ell \in \texttt{LINES}} \texttt{COST}_\ell x_\ell \\
\text{s.t.} \quad & \sum_{i \in \texttt{TRANSITS}} x_{(\texttt{USINE},i)} \leq \texttt{MAXPROD} \\
& \sum_{i \in \texttt{TRANSITS}} x_{(i,j)} \geq \texttt{DEMAND}_j, & \forall j \in \texttt{STORES} \\
& x_{(\texttt{USINE},i)} = \sum_{j \in \texttt{STORES}} x_{(i,j)}, & \forall i \in \texttt{TRANSITS} \\
& 0 \leq x_\ell \leq \texttt{CAPACITY}_\ell, & \forall \ell \in \texttt{LINES}.
\end{aligned}
$$

1.4 minimum distance (1-norm)

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum L^1 norm:

$$\|x\|_1 = \sum_{j=1,\dots,n} |x_j|$$

- variable splitting:

$$|x| = \min\{x^+ + x^- \mid x = x^+ - x^-, x^+, x^- \geq 0\}$$

$$\begin{aligned}
&\min \sum_{j=1}^{n}(x_j^+ + x_j^-) \\
&\text{s.t.}\;\; Ax = b, \\
&\qquad x_j = x_j^+ - x_j^-, && \forall j \\
&\qquad x_j^+, x_j^- \geq 0, && \forall j
\end{aligned}$$

- supporting plane model:

$$|x| = \max\{x, -x\} = \min\{y \mid y \geq x, y \geq -x\}$$

$$\begin{aligned}
&\min \sum_{j=1}^{n} y_j \\
&\text{s.t.}\;\; Ax = b, \\
&\qquad y_j \geq x_j, && \forall j \\
&\qquad y_j \geq -x_j, && \forall j
\end{aligned}$$

1.5 minimum distance (infinity-norm)

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum L^∞ norm:

$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j|$$

- $y \geq |x_j| \iff y \geq x_j \wedge y \geq -x_j$
- $y \geq \max_j |x_j| \iff y \geq x_j \wedge y \geq -x_j \; (\forall j)$

$$\begin{aligned}
&\min y \\
&\text{s.t.}\;\; Ax = b, \\
&\qquad y \geq x_j, && \forall j \\
&\qquad y \geq -x_j, && \forall j
\end{aligned}$$

1.6 data fitting (LAD regression)

Given m observations – data points $a_i \in \mathbb{R}^n$ and associate values $b_i \in \mathbb{R}$, $i = 1..m$ – predict the value of any point $a \in \mathbb{R}^n$ according to a linear regression model ?

A best **linear fit** is a function :

$$b(a) = a^T x + y, \text{ for chosen } x \in \mathbb{R}^n, y \in \mathbb{R}$$

minimizing the **residual/prediction error** $|b(a_i) - b_i|$, globally over the dataset $i = 1..m$, e.g:

Least Absolute Deviation or L_1-regression:

$$\min \sum_i |b(a_i) - b_i|$$

supporting planes

$$\begin{aligned} & min \sum_i d_i \\ \text{s.t. } & d_i \geq \sum_j a_{ij} x_j + y - b_i, && \forall i \\ & d_i \geq -(\sum_j a_{ij} x_j + y - b_i), && \forall i \\ & d \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

sparse supporting planes

$$\begin{aligned} & min \sum_i d_i \\ \text{s.t. } & r_i = \sum_j a_{ij} x_j + y - b_i, && \forall i \\ & d_i \geq r_i, && \forall i \\ & d_i \geq -r_i, && \forall i \\ & r, d \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

variable splitting

$$\begin{aligned} & min \sum_i d_i^+ + d_i^- \\ \text{s.t. } & d_i^+ - d_i^- = \sum_j a_{ij} x_j + y - b_i, && \forall i \\ & d_i^+, d_i^- \geq 0, && \forall i \\ & x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

dual model

$$\begin{aligned} & max \sum_i b_i z_i \\ \text{s.t. } & \sum_i a_{ij} z_i = 0, && \forall j \\ & \sum_i z_i = 0, \\ & z_i \in [-1, 1], && \forall i \end{aligned}$$

1.7 capacity planning

find a least cost electric power capacity expansion plan:

- planning horizon: the next $T \in \mathbb{N}$ years
- forecast demand (in MW): $d_t \geq 0$ for each year $t = 1, \dots, T$
- existing capacity (oil-fired plants, in MW): $e_t \geq 0$ available for each year t
- options for expanding capacities: (1) coal-fired plant and (2) nuclear plant
  - lifetime (in years): $l_j \in \mathbb{N}$, for each option $j = 1, 2$
  - capital cost (in euros/MW): c_{jt} to install capacity j operable from year t
  - political/safety measure: share of nuclear should never exceed 20% of available capacity
- decision variables, x_{jt}: installed capacity (in MW) of type $j = 1, 2$ starting at year $t = 1, \dots, T$
- constraints, each year: total capacity meets the demand + nuclear share
- implied variables, y_{jt} available capacity (in MW) $j = 1, 2$ for year t

$$
\begin{aligned}
\min \ & \sum_{t=1}^{T}\sum_{j=1}^{2} c_{jt}x_{jt} \\
\text{s.t. } \ & y_{jt} = \sum_{s=\max\{1,t-l_j+1\}}^{t} x_{js}, && \forall j=1,2,\ t=1,\ldots,T \\
& e_t + y_{1t} + y_{2t} \geq d_t, && \forall t=1,\ldots,T \\
& 8y_{2t} \leq 2e_t + 2y_{1t}, && \forall t=1,\ldots,T \\
& x_{jt} \geq 0, y_{jt} \geq 0, && \forall j=1,2,\ t=1,\ldots,T
\end{aligned}
$$