Mixed Integer Linear Programming
Course Notes on Modeling

OSE 2015: Optimization
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1 Model examples

1.1 Integer Knapsack Problem

Input: n items, value c_j and weight $w_j \geq 0$ for each item j, a capacity $K \geq 0$.
Output: a maximum value subset of items whose total weight does not exceed capacity K.

$$
\begin{aligned}
\max \; & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \; & \sum_{j=1}^{n} w_j x_j \leq K \\
& x_j \in \{0,1\} && j = 1..n
\end{aligned}
$$

with $x_j = 1$ iff item j is selected

1.2 Uncapacitated Facility Location Problem

Input: n facility locations, m customers, cost c_j to open facility j, cost d_{ij} to serve customer i from facility on location j.
Output: a minimum (opening and service) cost assignment of the customers to the open facilities.

$$
\begin{aligned}
\min \; & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n}\sum_{i=1}^{m} d_{ij} y_{ij} \\
\text{s.t.} \; & \sum_{j=1}^{n} y_{ij} = 1 && i = 1..m \\
& y_{ij} \leq x_j && j = 1..n,\; i = 1..m \\
& x_j \in \{0,1\} && j = 1..n \\
& y_{ij} \in \{0,1\} && j = 1..n,\; i = 1..m
\end{aligned}
$$

where $x_j = 1$ iff a facility is open at location j and $y_{ij} = 1$ iff customer i is served from facility j.

1.3 $1||C_{\max}$ Scheduling Problem

Input: n tasks and one machine, duration p_i for each task i.
Output: a minimum makespan schedule of the tasks on the machine.

$$
\begin{aligned}
& \min s_{n+1} && \\
& \text{s.t. } s_{n+1} \geq s_j + p_j && j = 1..n \\
& \quad s_j - s_i \geq M x_{ij} + (p_i - M) && i, j = 1..n \\
& \quad x_{ij} + x_{ji} = 1 && i, j = 1..n; i < j \\
& \quad s_j \in \mathbb{Z}_+ && j = 1..n+1 \\
& \quad x_{ij} \in \{0, 1\} && i, j = 1..n
\end{aligned}
$$

where $x_{ij} = 1$ iff task i precede task j, s_i is the starting time of task i, s_{n+1} is the makespan, and where $M \geq \sum_{i=1}^{n} p_i$.

1.4 Market Split Problem

Input: 1 company with 2 divisions, m products, n retailers, availability d_j for each product j, demand a_{ij} of each retailer i for each product j.
Output: an assignement of the retailers to the divisions approaching a 50/50 production split for each product.

$$
\begin{aligned}
& \min \sum_{j=1}^{m} s_j^+ + s_j^- && \\
& \text{s.t. } \sum_{i=1}^{n} a_{ij} x_i + s_j^+ - s_j^- = \frac{d_j}{2} && j = 1..m \\
& \quad x_i \in \{0, 1\} && i = 1..n \\
& \quad s_j^+ \geq 0, s_j^- \geq 0 && j = 1..m
\end{aligned}
$$

where $x_i = 1$ iff retailer i is assigned to division 1, $s_j^+ - s_j^-$ is the slack value (s_j^+ is the positive part and s_j^- is the negative part) between the volume produced by division 1 and the desired volume ($d_j * 50\%$).

1.5 Capacitated Transhipment Problem

Input: directed graph $G = (V, A)$, demand or supply b_i at each node n, capacity h_{ij} and unit flow cost c_{ij} on each arc (i, j).
Output: a minimum cost integer flow to satisfy the demand.

$$
\begin{aligned}
& \min \sum_{(i,j) \in A} c_{ij} x_{ij} && \\
& \text{s.t. } \sum_{j \in \delta^+(i)} x_{ij} - \sum_{j \in \delta^-(i)} x_{ij} = b_i && i \in V \\
& \quad x_{ij} \leq h_{ij} && (i, j) \in A \\
& \quad x_{ij} \in \mathbb{Z}_+ && (i, j) \in A
\end{aligned}
$$

where x_{ij} the flow on arc (i, j)

1.6 Traveling Salesman Problem

Input: a set V of cities, $E = V^2$, a distance $c_{ij} = c_{ji}$ between each cities i and j.
Output: a tour visiting every city exactly once.

$$
\begin{aligned}
\min \; & \sum_{e\in E} c_e x_e \\
\text{s.t.} \; & \sum_{e\in E | i\in e} x_e = 2 && i \in V \\
& \sum_{\delta(Q)} x_e \geq 2 && \emptyset \subsetneq Q \subsetneq V \\
& x_e \in \{0,1\} && e \in E
\end{aligned}
$$

where $x_e = 1$ iff the edge e belongs to the tour.

1.7 Uncapacitated Lot Sizing Problem

Input: n time periods, fix production cost f_t, unit production cost p_t, unit storage cost h_t at period t, demand d_t at each period t.
Output: a minimum (production and storage) cost production plan that satsify the demand.

$$
\begin{aligned}
\min \; & \sum_{t=1}^{n} f_t y_t + \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} h_t s_t \\
\text{s.t.} \; & s_{t-1} + x_t = d_t + s_t && t = 1..n \\
& x_t \leq M_t y_t && t = 1..n \\
& y_t \in \{0,1\} && t = 1..n \\
& s_t, x_t \geq 0 && t = 1,\ldots,n \\
& s_0 = 0
\end{aligned}
$$

where $y_t = 1$ iff production occurs during period t, x_t is the amount produced during period t, y_t is the amount stored at the beginning of period t, and where $M_t \geq \sum_{i=t}^{n} d_i$ for each period t.

$$
\begin{aligned}
\min \; & \sum_{t=1}^{n} f_t y_t + \sum_{i=1}^{n}\sum_{t=i}^{n} p_i z_{it} + \sum_{i=1}^{n}\sum_{t=i+1}^{n}\sum_{j=i}^{t-1} h_j z_{it} \\
\text{s.t.} \; & \sum_{i=1}^{t} z_{it} = d_t && t = 1..n \\
& z_{it} \leq d_t y_i && i = 1..n; t = i..n \\
& y_t \in \{0,1\} && t = 1..n \\
& z_{it} \geq 0 && i = 1..n; t = i..n
\end{aligned}
$$

where z_{it} is the amount produced in period i to satisfy demand of period t.

1.8 Bin Packing Problem

Input: n items, weight $w_j \geq 0$ for each item j, m containers each of capacity $K \geq 0$.
Output: an assignment of the items to a minimum number of containers.

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad & \sum_{j=1}^{m} w_j x_{ij} \leq K y_i && i = 1..n \\
& \sum_{i=1}^{n} x_{ij} = 1 && j = 1..m \\
& x_{ij} \in \{0,1\} && i = 1..n;\ j = 1..m \\
& y_i \in \{0,1\} && i = 1..n
\end{aligned}$$

where $y_i = 1$ iff container i is used and $x_{ij} = 1$ iff item j is assigned to container i.
The Dantzig-Wolfe formulation (can be solved by delayed column generation):

$$\begin{aligned}
\min \quad & \sum_{s \in \mathscr{S}} x_s \\
\text{s.t.} \quad & \sum_{s \in \mathscr{S}} a_{js} x_s = 1 && j = 1..n \\
& x_s \in \{0,1\} && s \in \mathscr{S}
\end{aligned}$$

where $\mathscr{S} = \{s \subset \{1,\dots,n\} \mid \sum_{j \in s} w_j \leq K\}$ is the set of all possible arrangements of items to one container, and $x_s = 1$ iff all the items in s (and no others) are assigned to the same container.

1.9 Multi 0-1 Knapsack Problem

Input: n items, value c_j and weight $w_j \geq 0$ for each item j, m containers, capacity $K_i \geq 0$ for each container i.
Output: a maximum value subset of items to assign to the containers such that the capacity of each container is not exceeded.

$$\begin{aligned}
\max \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_j x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} w_j x_{ij} \leq K_i && i = 1..m \\
& \sum_{i=1}^{m} x_{ij} \leq 1 && j = 1..n \\
& x_{ij} \in \{0,1\} && j = 1..n, i = 1..m
\end{aligned}$$

with $x_{ij} = 1$ iff item j is assigned to container i

The lagrangian dual:

$$\begin{aligned}
& \min z_\pi \\
& \text{s.t. } \pi_i \geq 0 \quad i = 1..m \\
& z_\pi = \quad \max \sum_{i=1}^{m} \sum_{j=1}^{n} c_j x_{ij} - \sum_{i=1}^{m} \pi_i \Big(\sum_{j=1}^{n} w_j x_{ij} - K_i\Big) \\
& \qquad \text{s.t. } \sum_{i=1}^{m} x_{ij} \leq 1 \qquad j = 1..n \\
& \qquad x_{ij} \in \{0,1\} \qquad j = 1..n, i = 1..m
\end{aligned}$$

where π_i is the penalty for violating the capacity of container i

2 Outline

2.1 Modeling booleans with binary variables

indicator	linearization
$\delta = 1 \iff x > 0, x \in \mathbb{Z}_+$	$\delta \leq x \leq U\delta$
$\delta = 1 \iff x > a$	$(a+\epsilon)\delta \leq x \leq a + (U-a)\delta$
$\delta = 1 \iff a \leq x < b$	need 2 indicators
$\delta = 1 \implies f \geq a$	$f \geq m + (a-m)\delta$
$\delta = 0 \implies f \geq a$	$f \geq m + (a-m)(1-\delta)$
$\delta = 1 \implies f < b$	$f \leq M + (b+\epsilon-M)\delta$
$f \geq b \implies \delta = 0$	$f \leq M + (b+\epsilon-M)\delta$

with $x \in [0, U] \subseteq \mathbb{R}_+$, $Ay \in [m, M] \subseteq \mathbb{R}$ and indicator $\delta \in \{0, 1\}$

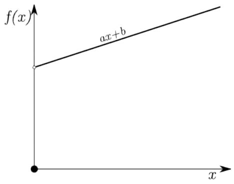
- It is often not necessary to enforce equivalence ($\iff$) between the indicator and the condition: one-sense implication ($\implies$) can be enough (and simplier)

2.2 Modeling logic/numeric relations with binary variables

condition	example	linearization
exclusive disjunction	*either c or* $\bar{c}$	$\delta = 1 \iff c$
exclusive disjunction	*either* c_1 *or* c_2	$\delta_1 + \delta_2 = 1$
disjunction	c_1 *or* c_2	$\delta_1 + \delta_2 \geq 1$
dependency	*if* c_1 *then* c_2	$\delta_2 \geq \delta_1$
exclusive alternative	*exactly* 1 *out of n*	$\sum_{i=1}^{n} \delta_i = 1$
counter	*exactly k out of n*	$\sum_{i=1}^{n} \delta_i = k$
bound	*at least k out of n*	$\sum_{i=1}^{n} \delta_i \geq k$
bound	*at most k out of n*	$\sum_{i=1}^{n} \delta_i \leq k$

2.3 Modeling non-linear functions with binary variables

2.3.1 set-up value:

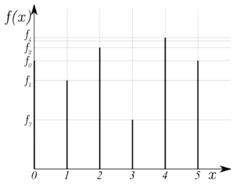


$f : [0, U] \subseteq \mathbb{R}_+ \to \mathbb{R}_+$

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ ax + b & \text{if } 0 < x \leq U \end{cases}$$

$f(x) = ax + b\delta$
$\epsilon\delta \leq x \leq U\delta$
$\delta \in \{0, 1\}$

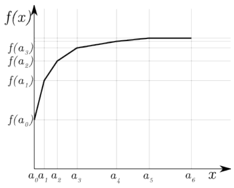
2.3.2 discrete value:



$f(x) = f_i$ if $x = i$

$f(x) = \sum_i \delta_i f_i$
$\sum_i i\delta_i = x$
$\sum_i \delta_i = 1$
$\delta_i \in \{0, 1\}$ $i = 0..n$

2.3.3 piecewise linear:



$f(x) = \sum_i \lambda_i f(a_i)$
$\sum_i a_i \lambda_i = x$
$\sum_i \lambda_i = 1$
$\lambda_i \in [0, 1]$ $i = 0..n$
with $SOS2(\lambda_i)$