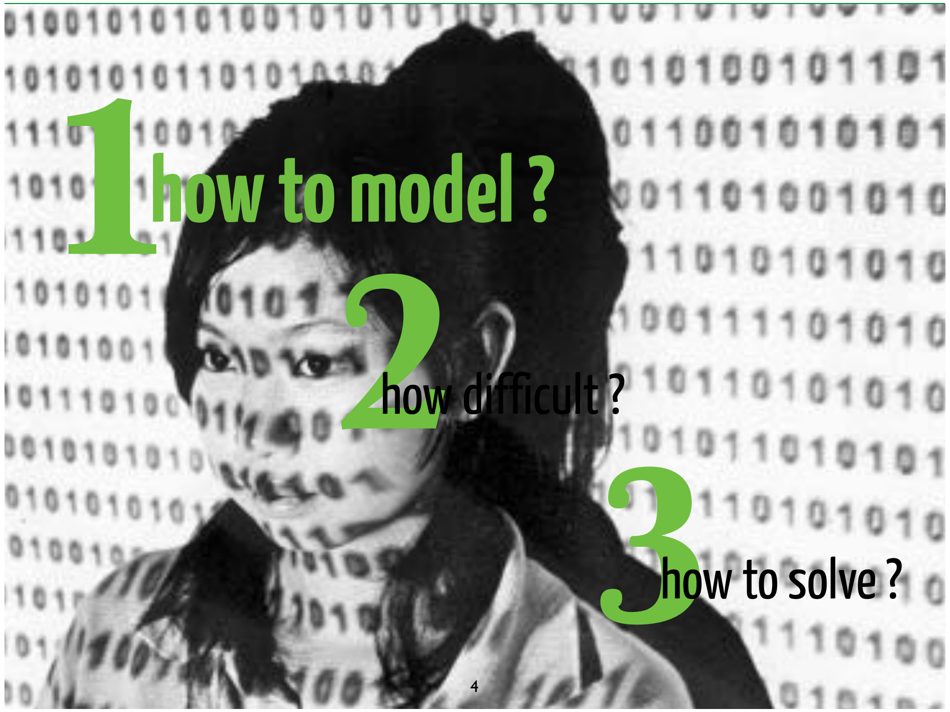
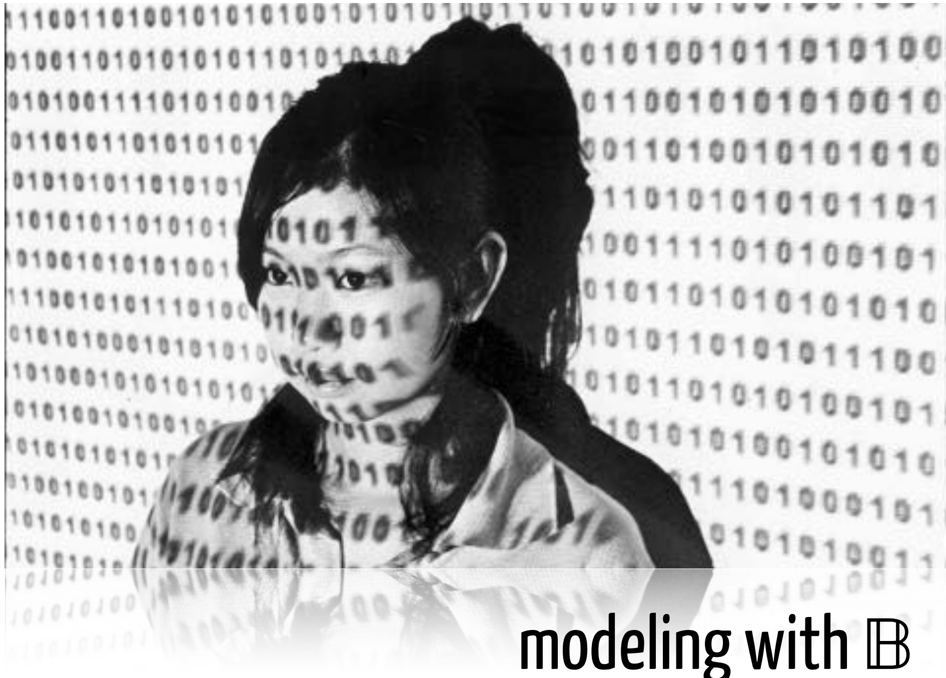
1 how to model ?

2 how difficult ?

3 how to solve ?





modeling with $\mathbb{B}$

~~true~~ 1 or ~~false~~ 0

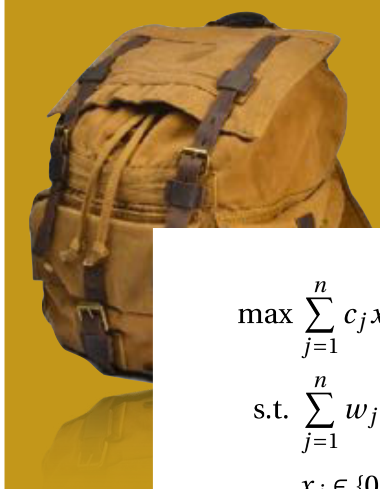
in an optimal solution...

- is item j selected ? $x_j \in \{0,1\}$
- is item j associated to item i ? $y_{ij} \in \{0,1\}$
- is non-negative x greater than a ? $x \geq ay,\ y \in \{0,1\}$
- is constraint c satisfied ? $\cdots$

Integer Knapsack Problem

Input n items, value c_j and weight w_j for each item j, capacity K.
Output a maximum value subset of items whose total weight does not exceed K.

x_j is item j packed ?



Integer Knapsack Problem

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} w_j x_j \leq K$$

$$x_j \in \{0,1\} \qquad j = 1..n$$

x_j is item j packed ?

logic with binaries

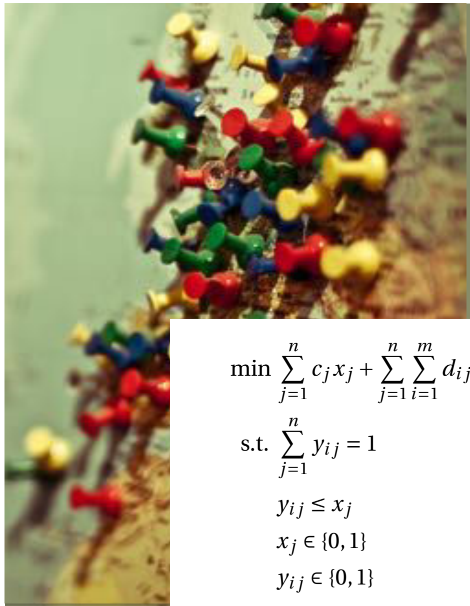
- either x or y $\quad x + y = 1$
- if x then y $\quad y \geq x$
- if x then f ≤ a $\quad f \leq ax + M(1-x)$ "big M"
- at most 1 out of n $\quad x_1 + \cdots + x_n \leq 1$
- at least k out of n $\quad x_1 + \cdots + x_n \geq k$

Uncapacitated Facility Location Problem

Input n facility locations, m customers, cost c_j to open facility j, cost d_{ij} to serve customer i from facility j

Output a mimimum (opening and service) cost assignment of customers to facilities.

x_j is location j open ? y_{ij} is customer i served from j ?



Uncapacitated Facility Location Problem

$$
\begin{aligned}
\min \; & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \sum_{i=1}^{m} d_{ij} y_{ij} & \\
\text{s.t.} \; & \sum_{j=1}^{n} y_{ij} = 1 & i = 1..m \\
& y_{ij} \leq x_j & j = 1..n,\ i = 1..m \\
& x_j \in \{0,1\} & j = 1..n \\
& y_{ij} \in \{0,1\} & j = 1..n,\ i = 1..m
\end{aligned}
$$

ons, m
en facility
tomer i from
g and
of

x_j is location j open ? y_{ij} is customer i served from j ?

1||Cmax Scheduling Problem

Input n tasks, duration pi for each task i, one machine

Output a minimal makespan schedule of the tasks on the machine without overlap

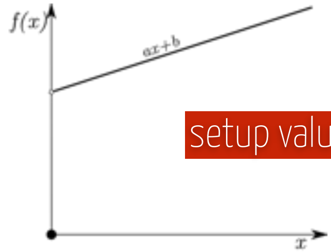
x_{ij} does i precede j ? s_j starting time of j

1||Cmax Scheduling Problem

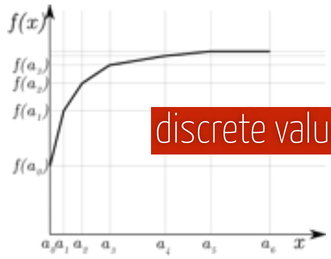
$$
\begin{array}{lll}
\min s_{n+1} & & \\
\text{s.t. } s_{n+1} \geq s_j + p_j & & j = 1..n \\
& s_j - s_i \geq M x_{ij} + (p_i - M) & i,j = 1..n \\
& x_{ij} + x_{ji} = 1 & i,j = 1..n; i < j \\
& s_j \in \mathbb{Z}_+ & j = 1..n+1 \\
& x_{ij} \in \{0,1\} & i,j = 1..n
\end{array}
$$

x_{ij} does i precede j ? s_j starting time of j

non-linear functions

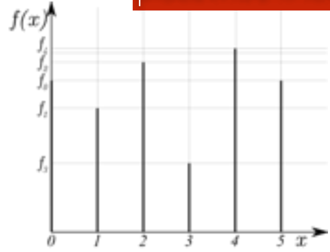


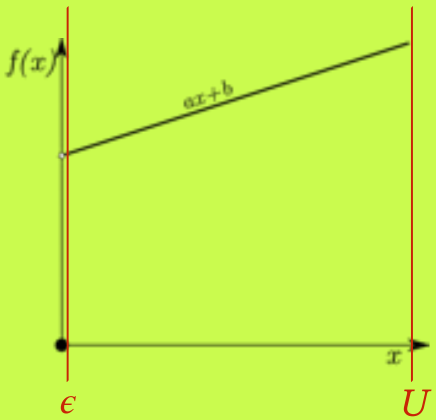
setup value



discrete values

piecewise linear

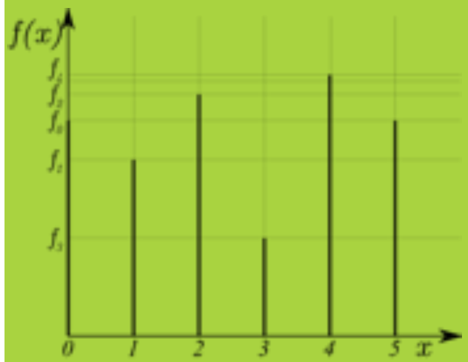




setup value

$$
\begin{aligned}
& f(x) = ax + b\delta \\
& \epsilon\delta \leq x \leq U\delta \\
& \delta \in \{0,1\}
\end{aligned}
$$

δ is x positive ?

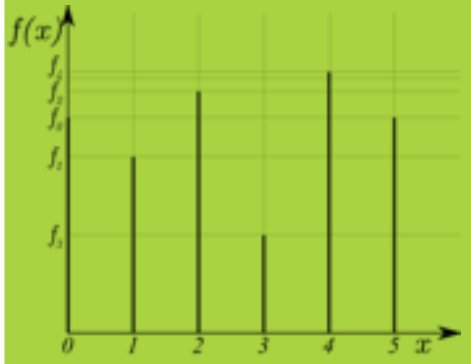


discrete values

$$
\begin{aligned}
& f(x) = \sum_i \delta_i f_i \\
& \sum_i i\delta_i = x \\
& \sum_i \delta_i = 1 \\
& \delta_i \in \{0,1\} \; i = 0..n
\end{aligned}
$$

δ_i is $x=i$ (and $f(x)=f_i$) ?

Special Ordered Set of type 1:
ordered set of variables, all zero except at most one



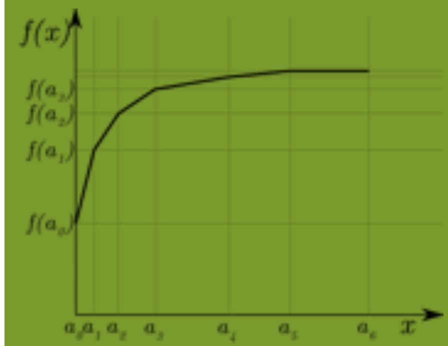
discrete values

$$f(x) = \sum_i \delta_i f_i$$
$$\sum_i i\delta_i = x$$
$$\sum_i \delta_i \geq 1$$
$$\delta_i \in \{0,1\}\ i = 0..n$$

SOS1(δ)

δ_i is $x=i$ (and $f(x)=f_i$) ?

Special Ordered Set of type 2:
ordered set of variables, all zero except at most two consecutive



piecewise linear

$$f(x) = \sum_i \lambda_i f(a_i)$$
$$\sum_i a_i \lambda_i = x$$
$$\sum_i \lambda_i = 1$$
$$\lambda_i \in [0,1]\ i = 0..n$$

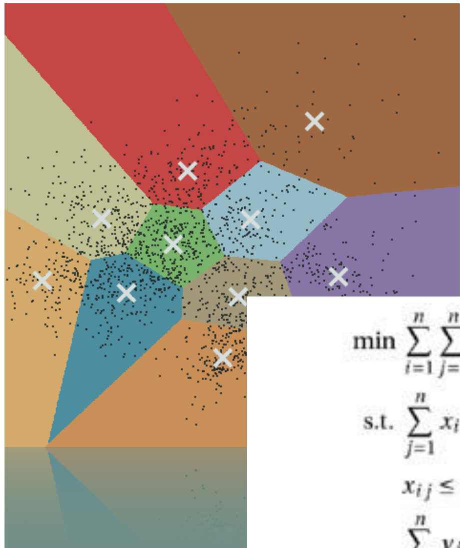
SOS2(λ)

λ_i is $x=a_i$? (then $\lambda_i a_i + \lambda_{i+1} a_{i+1}$ in $[a_i, a_{i+1}]$ if $\lambda_i + \lambda_{i+1} = 1$)

K-median clustering

Input n data points, distance d_{ij} between each two points i,j, number k of clusters.
Output k centers minimizing the sum of distances between each point and its nearest center.

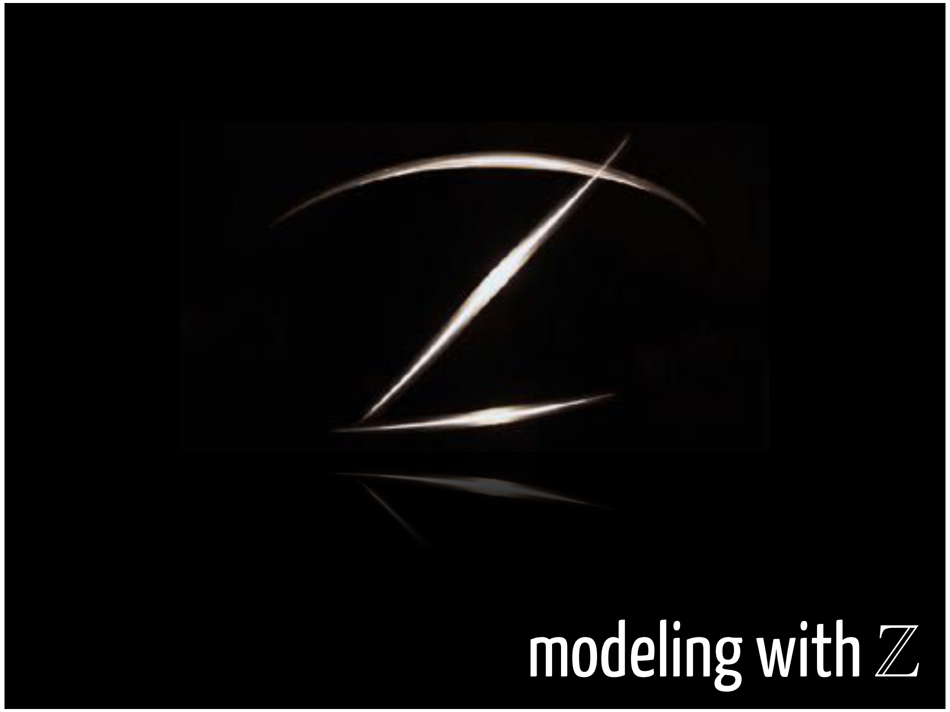
y_j is j a center ? x_{ij} is j the nearest center of i ?



K-median clustering

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij}$$
$$\text{s.t.} \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1..n$$
$$x_{ij} \leq y_j \qquad i,j = 1..n$$
$$\sum_{j=1}^{n} y_j = k$$
$$x_{ij} \in \{0,1\}, y_j \in \{0,1\} \qquad i,j = 1..n$$

y_j is j a center ? x_{ij} is j the nearest center of i ?



$$x_i = 5$$

to order i is the 5th item

to count 5 items are selected

to measure time task i starts at time 5

to measure space item i is located on floor 5

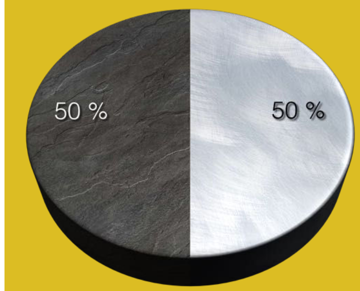
$$\simeq \delta_{i5} = 1$$

Binary Integer Linear Program (BIP) $\{0,1\}^n$

Integer Linear Program (IP) $\mathbb{Z}^n$

Mixed Integer Linear Program (MIP) $\mathbb{Z}^n \cup \mathbb{Q}^n$

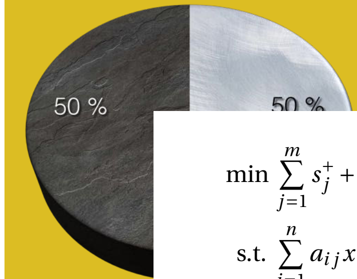
Interlude 1



Market Split Problem

Input 1 company, 2 divisions, m products with availabilities d_j, n retailers with demands a_{ij} in each product j.
Output an assignment of the retailers to the divisions approaching a 50/50 production split.

Interlude 1



Market Split Problem

$$\min \sum_{j=1}^{m} s_j^+ + s_j^-$$

$$\text{s.t.} \sum_{i=1}^{n} a_{ij} x_i + s_j^+ - s_j^- = \frac{d_j}{2} \qquad j = 1..m$$

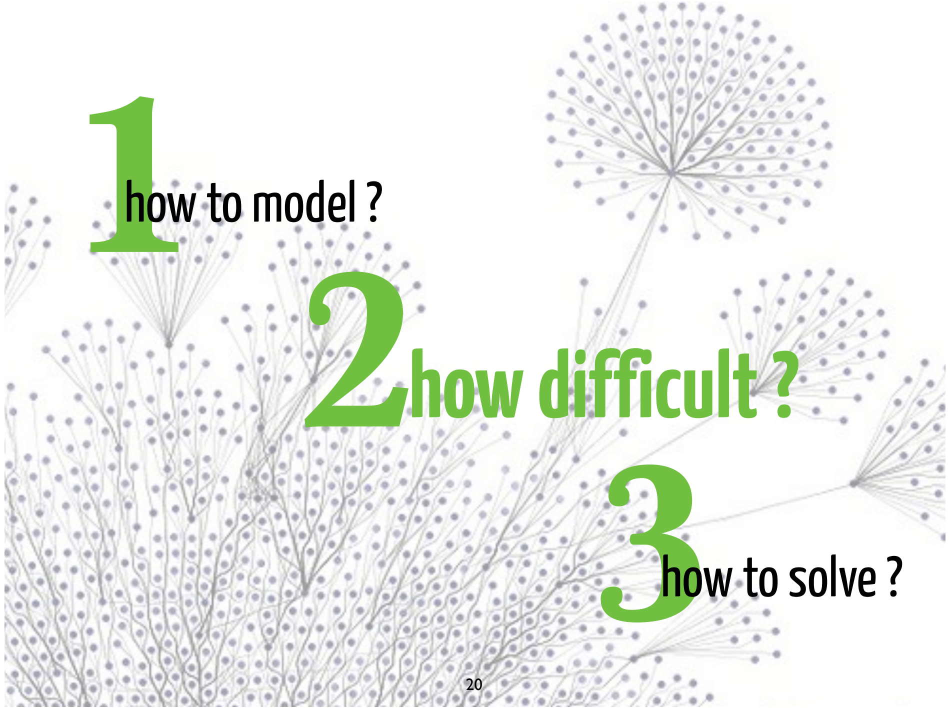
$$x_i \in \{0,1\} \qquad i = 1..n$$

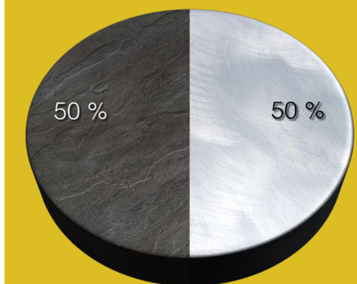
$$s_j^+ \geq 0, s_j^- \geq 0 \qquad j = 1..m$$

ns, m
es d_j, n
in each
e retailers
ng a 50/50

x_i is retailer i assigned to division 1 ?

s_j gap to the 50% split goal for product j





MIPLIB markshare_5_0

Input 5 products, 40 retailers

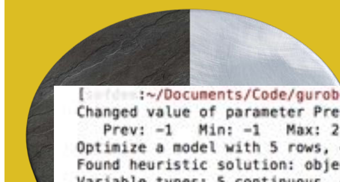
Output . . . .

. . . (hold the line please) . . . .

Int Opt = 1

Solution time = 20 minutes

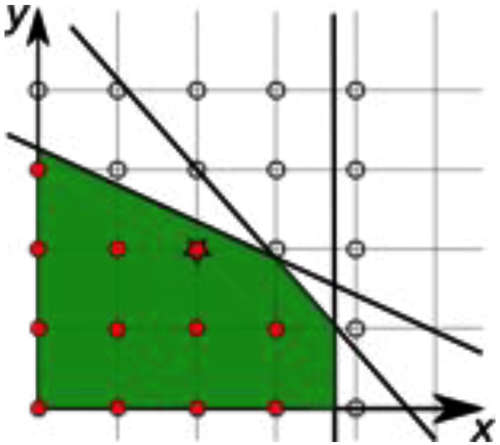
Proof time = > 1 hour



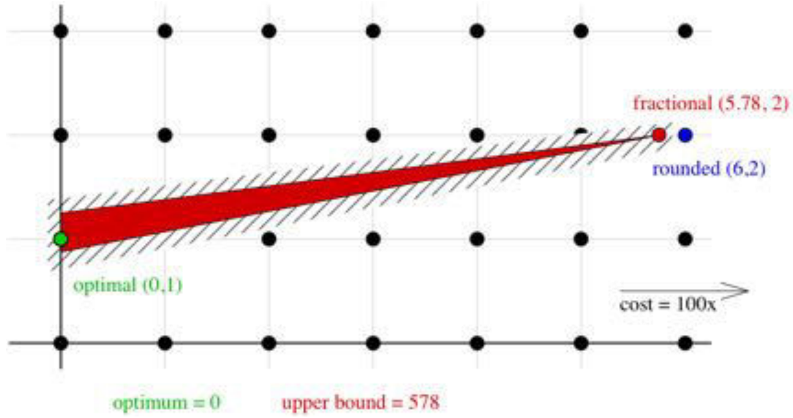
MIPLIB markshare_5_0

```
[[illegible]:~/Documents/Code/gurobi]$ gurobi.sh mymip.py markshare_5_0.mps.gz
Changed value of parameter Presolve to 0
   Prev: -1   Min: -1   Max: 2   Default: -1
Optimize a model with 5 rows, 45 columns and 203 nonzeros
Found heuristic solution: objective 5335
Variable types: 5 continuous, 40 integer (40 binary)

Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000  100%     -    0s

*62706364 28044              38       1.0000000    0.00000  100%   2.1 1241s

Explored 233848403 nodes (460515864 simplex iterations) in 3883.56 seconds
Thread count was 4 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.000000000000e+00, best bound 1.000000000000e+00, gap 0.0%
Optimal objective: 1
```

LP ≠ ILP



round LP ≠ ILP



“ILP is NP-hard: I can’t solve it !”

1||Cmax Scheduling Problem

$$
\begin{array}{lll}
\min & s_{n+1} \; = p_1 + \ldots + p_n & \\
\text{s.t.} & s_{n+1} \geq s_j + p_j & j = 1..n \\
 & s_j - s_i \geq M x_{ij} + (p_i - M) & i, j = 1..n \\
 & x_{ij} + x_{ji} = 1 & i, j = 1..n;\, i < j \\
 & s_j \in \cancel{\mathbb{Z}_+} \; \geq 0 & j = 1..n+1 \\
 & x_{ij} \in \{0, 1\} & i, j = 1..n
\end{array}
$$

Capacitated Transhipment Problem

Input digraph (V,A), demand or supply b_i at each node i, capacity h_{ij} and unit flow cost c_{ij} for each arc (i,j)

Output a mimimum cost integer flow to satisfy the demand

x_{ij} flow on arc (i,j)

Capacitated Transhipment Problem

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

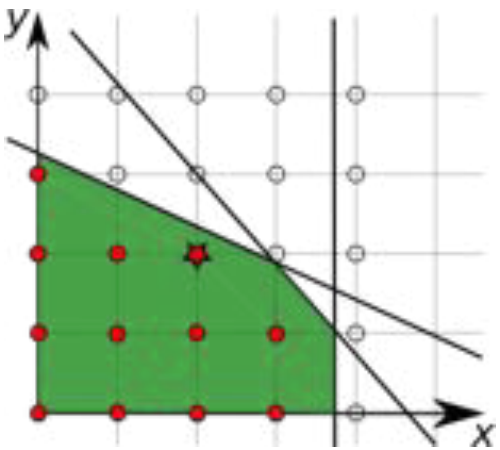
$$\text{s.t.} \sum_{j\in\delta^+(i)} x_{ij} - \sum_{j\in\delta^-(i)} x_{ij} = b_i \qquad i \in V$$

$$x_{ij} \leq h_{ij} \qquad (i,j) \in A$$

$$x_{ij} \in \cancel{\mathbb{Z}_+} \geq 0 \qquad (i,j) \in A$$

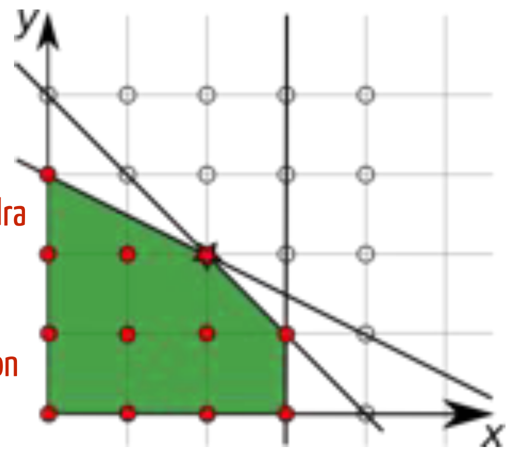
x_{ij} flow on arc (i,j)

LP = ILP



LP = ILP

integral polyhedra
=
convex hull
=
ideal formulation



totally unimodular matrix
(theory)

$$(P) = \max\{\ cx \mid Ax \leq b, x \in \mathbb{Z}^n_+\ \}$$

- basic feasible solutions of the LP relaxation $(\bar{P})$ take the form: $\bar{x} = (\bar{x}_B, \bar{x}_N) = (B^{-1}b, 0)$ where B is a square submatrix of (A, I_m)
- Cramer's rule: $B^{-1} = B^*/det(B)$ where B^* is the adjoint matrix (made of products of terms of B)
- Proposition: if (P) has integral data (A, b) and if $det(B) = \pm 1$ then $\bar{x}$ is integral

Definition

A matrix A is totally unimodular (TU) if every square submatrix has determinant $+1$, -1 or 0.

Proposition

If A is TU and b is integral then any optimal solution of $(\bar{P})$ is integral.

totally unimodular matrix
(practice)

How to recognize TU ?

Sufficient condition

A matrix A is TU if

- all the coefficients are $+1$, -1 or 0
- each column contains at most 2 non-zero coefficient
- there exists a partition (M_1, M_2) of the set M of rows such that each column j containing two non zero coefficients satisfies $\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0$.

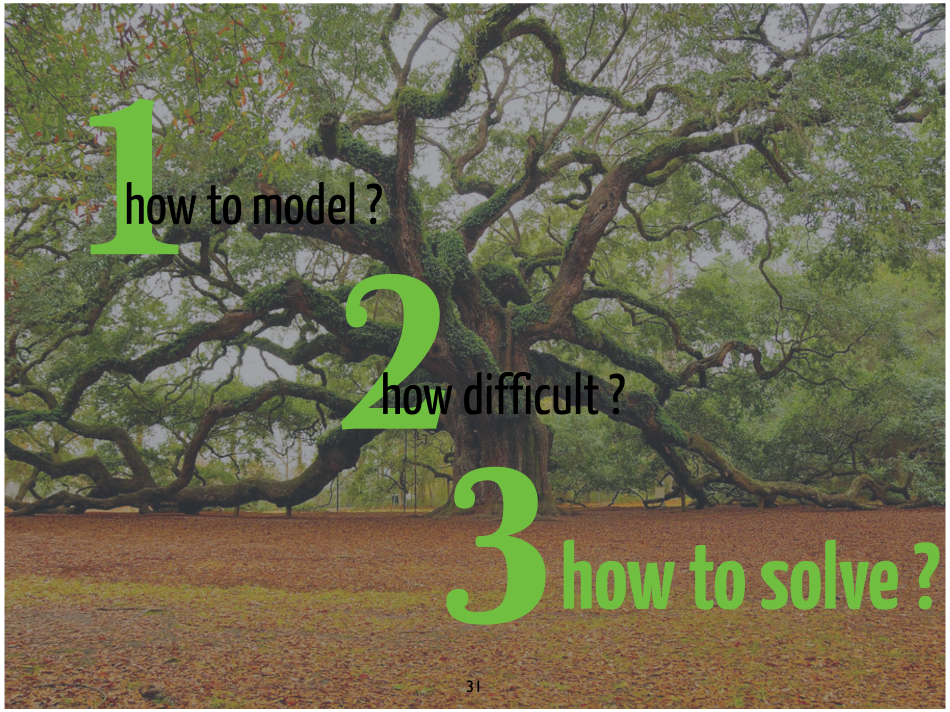
Proposition

A is TU $\iff$ A^t is TU $\iff$ (A, I_m) is TU
where A^t is the transpose matrix, I_m the identitiy matrix

Interlude 2

Show that the Transhipment ILP is ideal

Show that the Scheduling ILP is NOT ideal

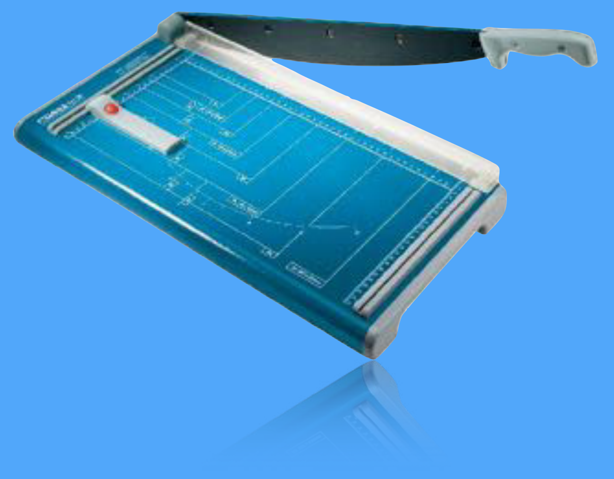


1 **Cuts** compute an ideal formulation and solve the LP

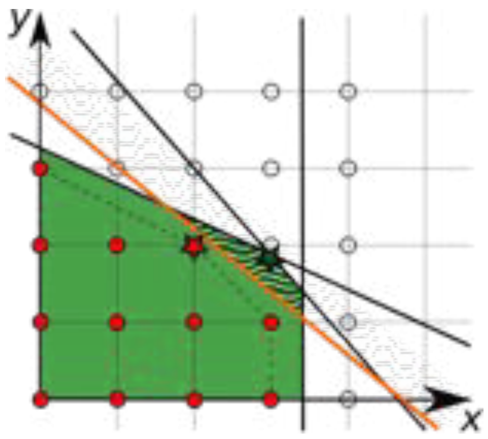
2 **Branch&Bound** enumerate solutions implicitely

3 **modern Branch&Cut** mix up+presolve +heuristics

4 **decomposition methods** (Branch&Price, Lagrangian, Benders)



Cutting Plane Algorithm



Cut valid inequality that separates the LP solution

Farkas Lemma any cut is a linear combination of the constraints

cutting plane algorithm

```
1. solve the LP relaxation (P), get x*
2. if x* is integral, STOP
3. find a cut for (P,x*) from a template T
```

templates

generic
Gomory Mixed Integer, Mixed Integer Rounding, Split, Chvátal-Gomory

structural
clique, cover, flow cover, zero half

problem-specific
subtour elimination (TSP), odd-set (matching)

ex 1 Mixed Integer Rounding

Combining constraints, then rounding leads to valid inequalities.

Let $u \in \mathbb{R}^m_+$, then the following inequalities are valid for (P):

- surrogate: $\sum_{j=1}^m u_j a_{ij} x_i \le \sum_{j=1}^m u_j b_j$ (since $u \ge 0$)
- round off: $\sum_{j=1}^m \lfloor u_j a_{ij} \rfloor x_i \le \sum_{j=1}^m u_j b_j$ (since $\lfloor u_j a_{ij} \rfloor \le u_j a_{ij}$ and $x \ge 0$)
- Chvátal-Gomory: $\sum_{j=1}^m \lfloor u_j a_{ij} \rfloor x_i \le \lfloor \sum_{j=1}^m u_j b_j \rfloor$ (since $e \in \mathbb{Z}$ and $e \le f$ implies that $e \le \lfloor f \rfloor$)

- CG inequalities form a generic class of valid inequalities: they apply to any IP

ex 2 Cover

Cover inequalities

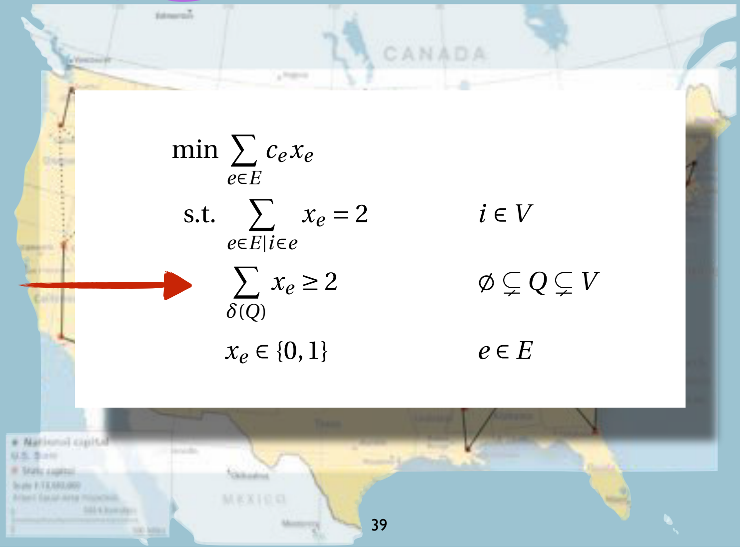
$$S = \{y \in \{0,1\}^7 | 11y_1 + 6y_2 + 6y_3 + 5y_4 + 5y_5 + 4y_6 + y_7 \le 19\}$$

- (y_3, y_4, y_5, y_6) is a minimal cover for $11y_1 + 6y_2 + 6y_3 + 5y_4 + 5y_5 + 4y_6 + y_7 \le 19$ as $6+5+5+4 > 19$ then $y_3 + y_4 + y_5 + y_6 \le 3$ is a cover inequality
- we can derive a stronger valid inequality $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \le 3$ by noting that y_1, y_2 has greater coefficients than any variable in the cover
- note furthermore that (y_1, y_i, y_j) is a cover $\forall i \neq j \in \{2,3,4,5,6\}$ then $2y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \le 3$ is also valid

The procedure to get this last equality is called *lifting*

ex 3 Subtour for TSP

ex 3 Subtour for TSP



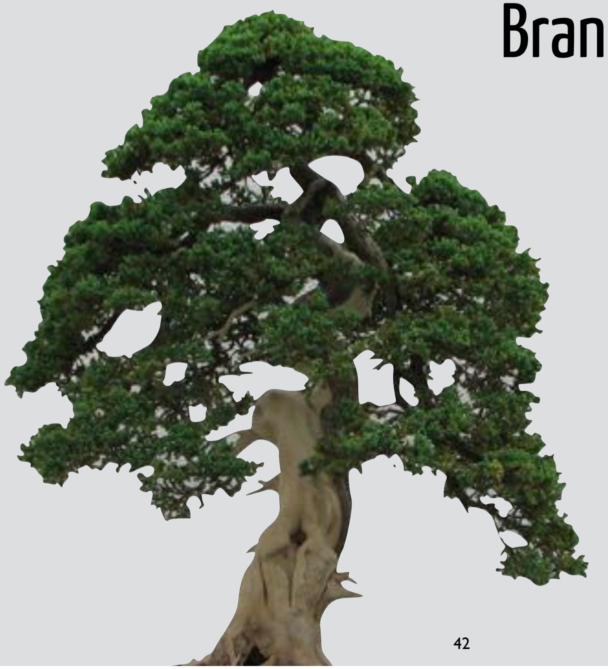
$$
\begin{aligned}
\min \ & \sum_{e \in E} c_e x_e \\
\text{s.t.} \ & \sum_{e \in E | i \in e} x_e = 2 && i \in V \\
& \sum_{\delta(Q)} x_e \geq 2 && \emptyset \subsetneq Q \subsetneq V \\
& x_e \in \{0,1\} && e \in E
\end{aligned}
$$

ex 3 Subtour for TSP

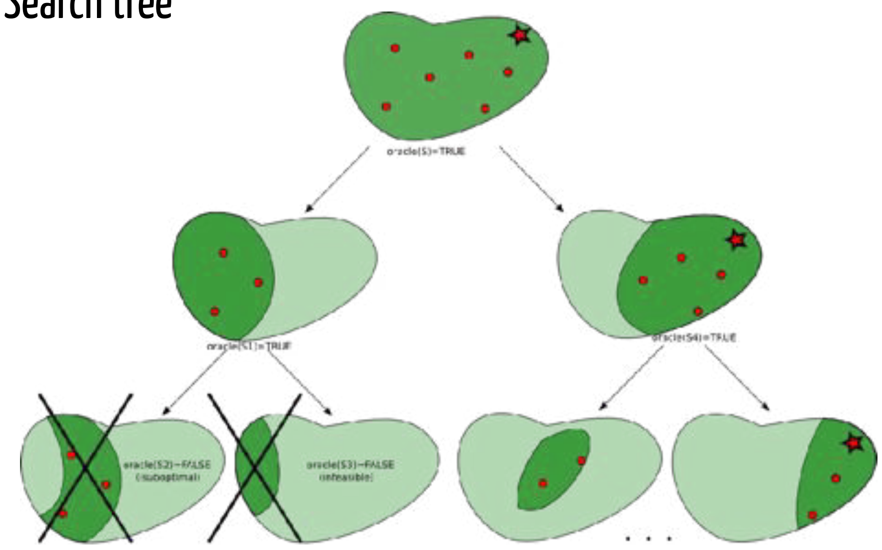
limits depending on the cut families

- the algorithm may stop prematurely
- the algorithm may not converge
- the algorithm may converge slowly
- the separation procedure may be NP-hard
- the LP grows
- the LP structure changes

Branch and Bound



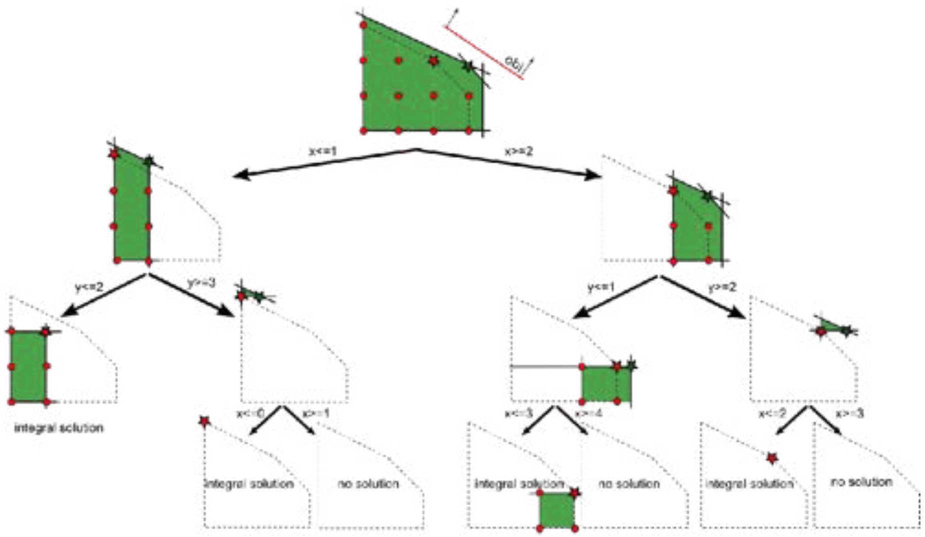
Search tree



LP-based B&B

```
oracle(S) = FALSE iff either:
 — LP is infeasible
 — the fractional solution x̄
   is not better than the
   incumbent x*
 — x̄ is integer (update x*)
```

then prune node S



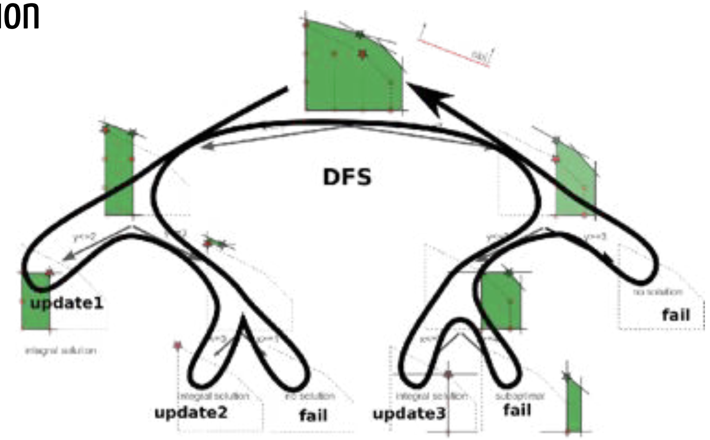
branching

node selection
which order to visit nodes ?

variable selection
how to separate nodes ?

constraint branching
alternative to variable branching

node selection

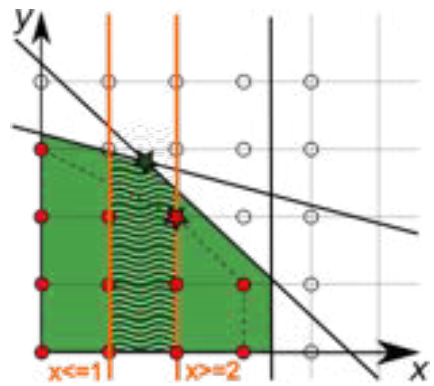


Best Bound First Search explore less nodes, manages larger trees

Depth First Search sensible to bad decisions at or near the root

DFS (up to n solutions) **+ BFS** (to prove optimality)

variable selection



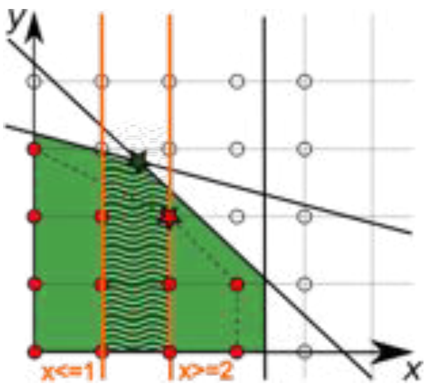
most fractional easy to implement but not better than random

strong branching best improvement among all candidates (impractical)

pseudocost branching record previous branching success for each var (inaccurate at root)

reliability branching pseudocosts initialised with strong branching

variable selection



most fractional easy to implement but not better than random

strong branching best improvement among all candidates (impractical)

pseudocost branching record previous branching success for each var (inaccurate at root)

reliability branching pseudocosts initialised with strong branching

constraint branching

example: GUB dichotomy

- if (P) contains a GUB constraint $\sum_C x_i = 1, x \in \{0,1\}^n$
- choose $C' \subseteq C$ s.t. $0 < \sum_{C'} \bar{x}_i < 1$
- create two child nodes by setting either $\sum_{C'} x_i = 0$ or $\sum_{C'} x_i = 1$

- enforced by fixing the variable values
- leads to more balanced search trees

SOS1 branching in a facility location problem

choose a warehouse depending on its size/cost:

$$\begin{aligned} \text{COST} &= 100x_1 + 180x_2 + 320x_3 + 450x_4 + 600x_5 \\ \text{SIZE} &= 10x_1 + 20x_2 + 40x_3 + 60x_4 + 80x_5 \\ \text{(SOS1)} &: x_1 + x_2 + x_3 + x_4 + x_5 = 1 \end{aligned}$$

- let $\bar{x}_1 = 0.35$ and $\bar{x}_5 = 0.65$ in the LP solution then SIZE$= 55.5$
- choose $C' = \{1,2,3\}$ in order to model SIZE≤ 40 or SIZE≥ 60

modern solvers

Simplex

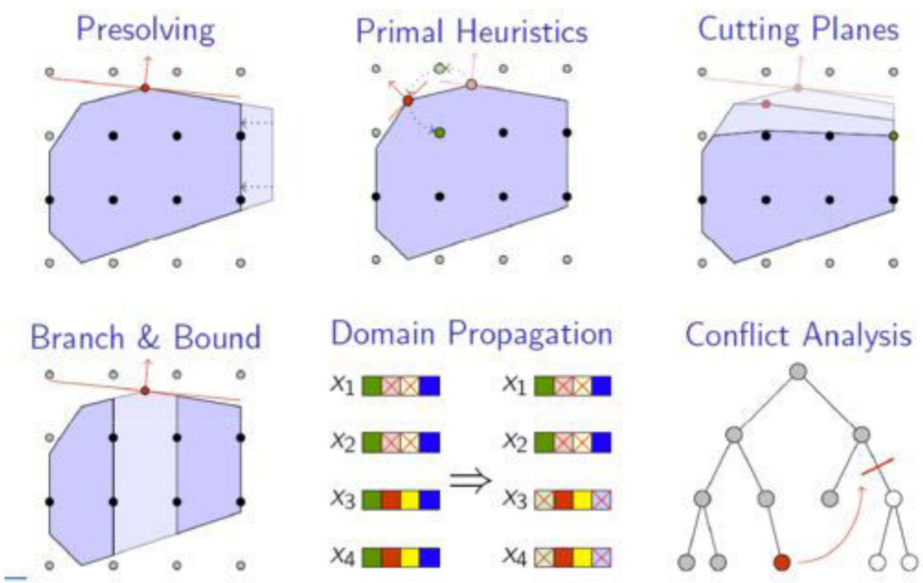
var branching

Preprocessing

Branch & Cut

Heuristics

Parallelism

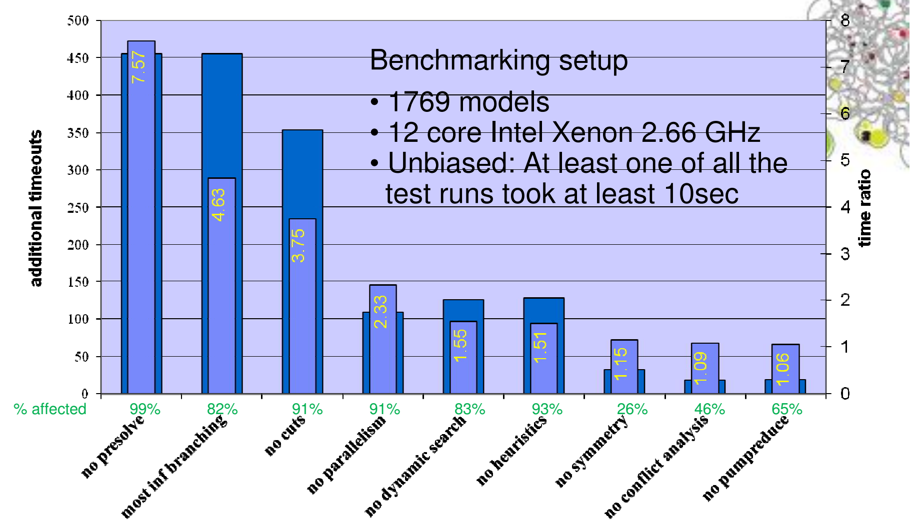


Slide from Martin Grötschel Co@W Berlin 2015

SmarterCommerce

IBM

Component Impact CPLEX 12.5 Summary



CPLEX 12.7

- Boolean Quadric Polytope (BQP) cuts
- Clique cuts
- Cover cuts
- Disjunctive cuts
- Flow cover cuts
- Flow path cuts
- Gomory fractional cuts
- Generalized upper bound (GUB) cover cuts
- Implied bound cuts: global and local
- Lift-and-project cuts
- Mixed integer rounding (MIR) cuts
- Multi-commodity flow (MCF) cuts
- Reformulation Linearization Technique (RLT) cuts
- Zero-half cuts

GUROBI 7.5

CliqueCuts	Clique cut generation
CoverCuts	Cover cut generation
FlowCoverCuts	Flow cover cut generation
FlowPathCuts	Flow path cut generation
GUBCoverCuts	GUB cover cut generation
ImpliedCuts	Implied bound cut generation
MIPSepCuts	MIP separation cut generation
MIRCuts	MIR cut generation
StrongCGCuts	Strong-CG cut generation
ModKCuts	Mod-k cut generation
NetworkCuts	Network cut generation
ProjImpliedCuts	Projected implied bound cut generation
SubMIPCuts	Sub-MIP cut generation
ZeroHalfCuts	Zero-half cut generation
InfProofCuts	Infeasibility proof cut generation

Preprocessing

reduce size

remove redundancies $x+y\leq 3$, *binaries*

substitute variables $x+y-z=0$

fix variables by duality $c_j\geq 0,\ A_j\geq 0 \Rightarrow x=x_{min}$

fix variables by probing $x=1$ *infeas* $\Rightarrow x=0$

strengthen LP relaxation

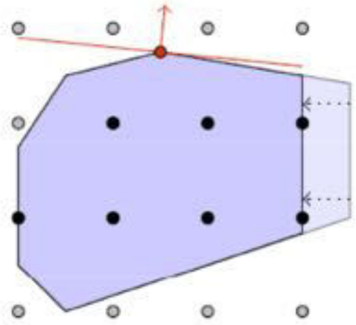
adjust bounds $2x+y\leq 1$, *binaries* $\Rightarrow x=0$

lift coefficients $2x-y\leq 1$, *binaries* $\Rightarrow x-y\leq 1$

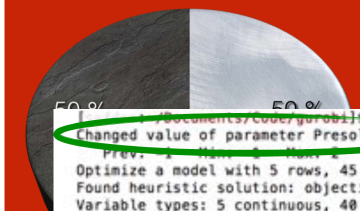
identify/exploit properties

detect implied integer $3x+y=7,\ x$ *int* $\Rightarrow y$ *int*

build the conflict graph
detect disconnected components
remove symmetries



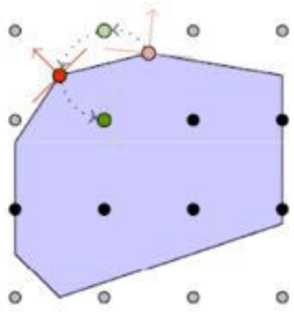
MIPLIB markshare_5_0




```
[     :~/Documents/Code/gurobi]$ gurobi.sh mymip.py markshare_5_0.mps.gz
Changed value of parameter Presolve to 0
   Prev: -1  Min: 0  Max: 2  Default: -1
Optimize a model with 5 rows, 45 columns and 203 nonzeros
Found heuristic solution: objective 5335
Variable types: 5 continuous, 40 integer (40 binary)

Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s

*62706364 28044              38       1.0000000    0.00000   100%   2.1 1241s

Explored 233848403 nodes (460515864 simplex iterations) in 3883.5 seconds
Thread count was 4 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.000000000000e+00, best bound 1.000000000000e+00, gap 0.0%
Optimal objective: 1
```



```
[sofdem:~/Documents/Code/gurobi]$ gurobi.sh mymip.py markshare_5_0.mps.gz
Optimize a model with 5 rows, 45 columns and 203 nonzeros
Found heuristic solution: objective 5335
Presolve time: 0.00s
Presolved: 5 rows, 45 columns, 203 nonzeros
Variable types: 0 continuous, 45 integer (40 binary)

Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                     320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                     239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29      96.0000000    0.00000   100%   2.7    0s
*   99    32              34      58.0000000    0.00000   100%   2.1    0s
H  506   214                      53.0000000    0.00000   100%   1.9    0s
H30682   442                       1.0000000    1.00000  0.00%   2.1    0s

Cutting planes:
  Cover: 26

Explored 30682 nodes (65348 simplex iterations) in 0.70 seconds
Thread count was 4 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.000000000000e+00, best bound 1.000000000000e+00, gap 0.0%
Optimal objective: 1
```



rounding LP solution
diving at some nodes
local search in the incumbent neighbourhood

Primal Heuristics

accelerate the search a little
appeal to the practitioner a lot

limits

- highly heuristic (branching decisions, cut generation)
- floating-point errors and optimality tolerance (0.01%)
- generic features
- less effective on general integers (ex: scheduling)
- hard to model (and solve) non-linear structures
- NP-hard

how to tune modern solvers

play with Gurobi


```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000  100%     -    0s
H    0     0                     320.0000000    0.00000  100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000  100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000  100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000  100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000  100%     -    0s
H    0     0                     239.0000000    0.00000  100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000  100%     -    0s
*   36     0              29      96.0000000    0.00000  100%   2.7    0s
*   99    32              34      58.0000000    0.00000  100%   2.1    0s
H  506   214                      53.0000000    0.00000  100%   1.9    0s
H30682   442                       1.0000000    1.00000 0.00%   2.1    0s
```

use as a heuristic

set a time limit
`MIPFocus=1`
`ImproveStartGap=0.1`

```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                     320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                     239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29      96.0000000    0.00000   100%   2.7    0s
*   99    32              34      58.0000000    0.00000   100%   2.1    0s
H  506   214                      53.0000000    0.00000   100%   1.9    0s
H30682   442                       1.0000000    1.00000  0.00%   2.1    0s
```

change the LP solver

if nbIteration(node) ≥ nbIteration(root)/2

`NodeMethod=2`


```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                     320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                     239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29      96.0000000    0.00000   100%   2.7    0s
*   99    32              34      58.0000000    0.00000   100%   2.1    0s
H  506   214                      53.0000000    0.00000   100%   1.9    0s
H30682   442                       1.0000000    1.00000  0.00%   2.1    0s
```


supply a feasible solution

if built-in heuristics fail

```
PumpPasses,MinRelNodes,ZeroObjNodes
model.read('initSol.mst')
model.cbSetSolution(vars, newSol)
```



```
Root relaxation: objective 0.000000e+00, 15 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.00000    0    5 5335.00000    0.00000   100%     -    0s
H    0     0                     320.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    6  320.00000    0.00000   100%     -    0s
     0     0    0.00000    0    5  320.00000    0.00000   100%     -    0s
H    0     0                     239.0000000    0.00000   100%     -    0s
     0     0    0.00000    0    5  239.00000    0.00000   100%     -    0s
*   36     0              29      96.0000000    0.00000   100%   2.7    0s
*   99    32              34      58.0000000    0.00000   100%   2.1    0s
H  506   214                      53.0000000    0.00000   100%   1.9    0s
H30682   442                       1.0000000    1.00000  0.00%   2.1    0s
```


tighten the model

if the bound stagnates

```
Cuts=3
Presolve=3
model.cbCut(lhs, sense, rhs)
```

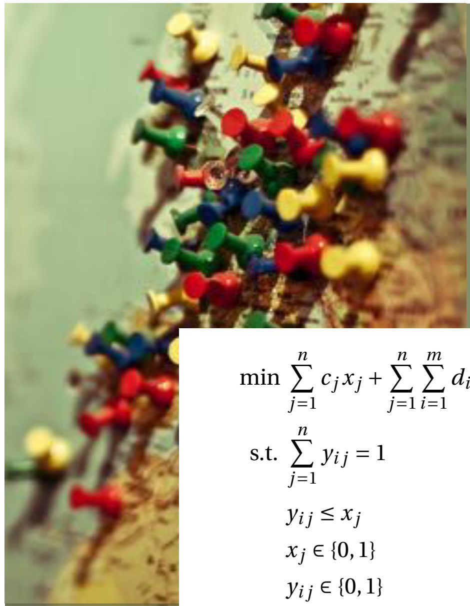

http://www.gurobi.com/

/documentation/8.0/refman/mip_models

/resources/seminars-and-videos

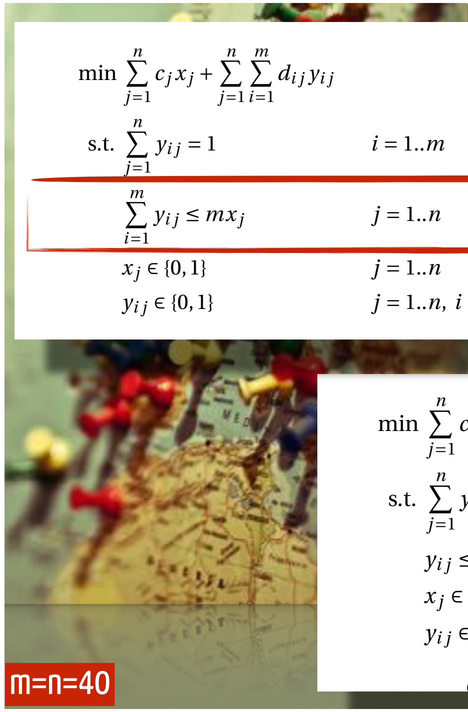
you know your problem better than your solver does

improve your model



Uncapacitated Facility Location Problem

$$
\begin{array}{lll}
\min & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \sum_{i=1}^{m} d_{ij} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} y_{ij} = 1 & i = 1..m \\
& y_{ij} \le x_j & j = 1..n,\ i = 1..m \\
& x_j \in \{0,1\} & j = 1..n \\
& y_{ij} \in \{0,1\} & j = 1..n,\ i = 1..m
\end{array}
$$



14 hours

$$
\begin{array}{lll}
\min & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \sum_{i=1}^{m} d_{ij} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} y_{ij} = 1 & i = 1..m \\
& \boxed{\sum_{i=1}^{m} y_{ij} \le m x_j} & j = 1..n \\
& x_j \in \{0,1\} & j = 1..n \\
& y_{ij} \in \{0,1\} & j = 1..n,\ i = 1..m
\end{array}
$$

Input n facility locations, m

2 seconds

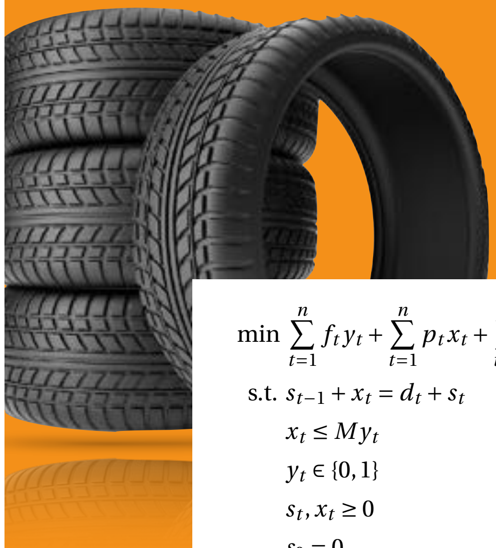
$$
\begin{array}{lll}
\min & \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \sum_{i=1}^{m} d_{ij} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} y_{ij} = 1 & i = 1..m \\
& y_{ij} \le x_j & j = 1..n,\ i = 1..m \\
& x_j \in \{0,1\} & j = 1..n \\
& y_{ij} \in \{0,1\} & j = 1..n,\ i = 1..m
\end{array}
$$

m=n=40

Uncapacitated Lot Sizing Problem

Input n time periods, fixed production cost f_t, unit production cost p_t, unit storage cost h_t, demand d_t for each period t

Output a mimimum (production and storage) cost production plan to satisfy the demand



Uncapacitated Lot Sizing Problem

$$\begin{aligned}
\min \; & \sum_{t=1}^{n} f_t y_t + \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} h_t s_t \\
\text{s.t. } & s_{t-1} + x_t = d_t + s_t && t = 1..n \\
& x_t \le M y_t && t = 1..n \\
& y_t \in \{0,1\} && t = 1..n \\
& s_t, x_t \ge 0 && t = 1,\dots,n \\
& s_0 = 0
\end{aligned}$$

z_{it} production in period i to satisfy demand of period t

Uncapacitated Lot Sizing Problem

$$\begin{aligned}
\min \; & \sum_{t=1}^{n} f_t y_t + \sum_{i=1}^{n} \sum_{t=i}^{n} p_i z_{it} + \sum_{i=1}^{n} \sum_{t=i+1}^{n} \sum_{j=i}^{t-1} h_j z_{it} \\
\text{s.t. } & \sum_{i=1}^{t} z_{it} = d_t && t = 1..n \\
& z_{it} \le d_t y_i && i = 1..n; t = i..n \\
& y_t \in \{0,1\} && t = 1..n \\
& z_{it} \ge 0 && i = 1..n; t = i..n
\end{aligned}$$

z_{it} production in period i to satisfy demand of period t

Uncapacitated Lot Sizing Problem

$$\begin{aligned}
\min \; & \sum_{t=1}^{n} f_t y_t + \sum_{i=1}^{n} \sum_{t=i}^{n} p_i z_{it} + \sum_{i=1}^{n} \sum_{t=i+1}^{n} \sum_{j=i}^{t-1} h_j z_{it} \\
\text{s.t. } & \sum_{i=1}^{t} z_{it} = d_t && t = 1..n \\
& z_{it} \le d_t y_i && i = 1..n; t = i..n \\
& y_t \in \{0,1\} && t = 1..n \\
& z_{it} \ge 0 && i = 1..n; t = i..n
\end{aligned}$$

LP=ILP

z_{it} production in period i to satisfy demand of period t

Bin Packing Problem

Input n containers, m items,
capacity c for all containers,
weight w_j for each item j
Output a packing of all items in
a mimimum number of containers

Bin Packing Problem

$$
\begin{aligned}
\min \; & \sum_{i=1}^{n} y_i & & \\
\text{s.t.} \; & \sum_{j=1}^{m} w_j x_{ij} \le c y_i & & i = 1..n \\
& \sum_{i=1}^{n} x_{ij} = 1 & & j = 1..m \\
& x_{ij} \in \{0,1\} & & i = 1..n;\; j = 1..m \\
& y_i \in \{0,1\} & & i = 1..n
\end{aligned}
$$

$\mathscr{S}$ all the possible arrangements of items in a bin

Bin Packing Problem

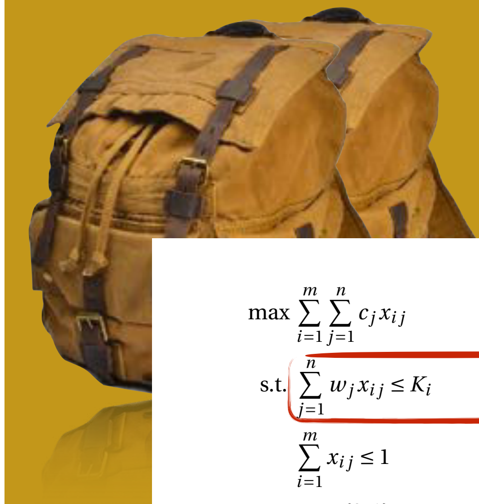
$$
\begin{aligned}
\min \; & \sum_{s \in \mathscr{S}} x_s & & \\
\text{s.t.} \; & \sum_{s \in \mathscr{S}} a_{js} x_s = 1 & & j = 1..n \\
& x_s \in \{0,1\} & & s \in \mathscr{S}
\end{aligned}
$$

delayed column generation

$\mathscr{S}$ all the possible arrangements of items in a bin

Multi 0-1 Knapsack Problem

Input n items, m bins, value c_j
and weight w_j for each item j,
capacity K_i for each bin i.
Output a maximum value subset of
items packed in the bins.



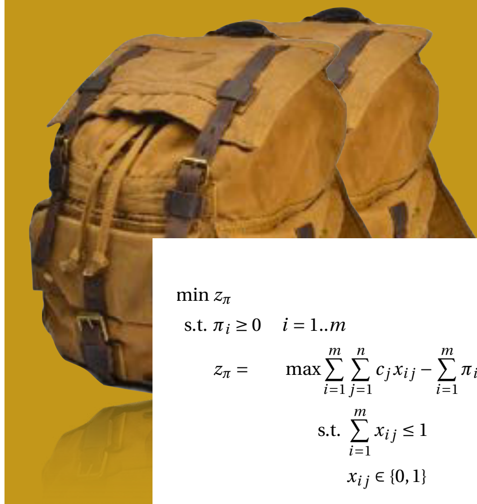
Multi 0-1 Knapsack Problem

$$\max \sum_{i=1}^{m}\sum_{j=1}^{n} c_j x_{ij}$$

$$\text{s.t.} \quad \boxed{\sum_{j=1}^{n} w_j x_{ij} \le K_i \qquad i = 1..m}$$

$$\sum_{i=1}^{m} x_{ij} \le 1 \qquad j = 1..n$$

$$x_{ij} \in \{0,1\} \qquad j = 1..n, i = 1..m$$



Multi 0-1 Knapsack Problem

$$\min z_\pi$$

$$\text{s.t. } \pi_i \ge 0 \quad i = 1..m$$

$$z_\pi = \quad \max \sum_{i=1}^{m}\sum_{j=1}^{n} c_j x_{ij} - \sum_{i=1}^{m} \pi_i\left(\sum_{j=1}^{n} w_j x_{ij} - K_i\right)$$

$$\text{s.t.} \sum_{i=1}^{m} x_{ij} \le 1 \qquad j = 1..n$$

$$x_{ij} \in \{0,1\} \qquad j = 1..n, i = 1..m$$

lagrangian relaxation

performance

maintainability

MIP advantages

transparency

extensibility

constraint programming

non-linear programming

but if all else fails

SAT

metaheuristics

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