Thesis Proposal:
# Mathematical programming with equilibrium constraints: models and algorithms for nonconvex optimization

A general form of *mathematical programs with equilibrium constraints* (MPECs) is as follows, with $f$ and $g$ continuously differentiable on a real $n$-space:

$$\min \quad f(x,z,w) \tag{1a}$$
$$\text{s.t.} \quad g(x,z,w) = 0 \tag{1b}$$
$$z \geq 0,\ w \geq 0,\ z^T w = 0. \tag{1c}$$

Constraints (1c) together with mapping $g$ are referred to as *complementarity restrictions* or *variational inequalities* (parametrized by $x$), and the term *equilibrium* refers to certain phenomena of balance that frequently manifest in a wide range of practical applications in engineering [5] and economy [6]. As a matter of fact, an equilibrium corresponds, both in physics or in game theory, to a minimum energy state, and Karush-Kuhn-Tucker (KKT) optimality conditions translate to complementary constraints. As such, MPECs appear as KKT reformulations of *bilevel programs* [4], also known as mathematical programs with optimization constraints, with upper-level variables $x$ and lower-level primal-dual pairs $z$ and $w$. Certain classes of nonconvex optimization problems, like the standard quadratic program [3] or in supervised classification [4], are also examined through their KKT reformulations as MPECs. Finally, MPEC can be viewed as a special issue of a *disjunctive program* (the quadratic equality in (1c) is equivalent to $z_i = 0$ OR $w_i = 0$ for all $i$), showing off its combinatorial nature.

All in all, MPEC is suitable for numerous applications, and it has thus been the subject of many studies. Most of the literature tackles the case with $f$ and $g$ being linear functions. Even this simplest case is challenging to solve, as, with no further restriction, the feasible set is nonconvex and lacks the regularity of standard constraint qualifications. The existing solution approaches seek to exploit the specific structure of the complementary constraints to obtain local or global optimality certificates or to accelerate the convergence of numerical algorithms to stationary points. While some are theoretically all-purpose MPEC solvers, these algorithms apply efficiently, in practice, only to specific contexts. Thus, there is an ongoing research into more robust algorithms. In particular, large-scale MPECs are computationally demanding due to their combinatorial nature and call for more scalable algorithms.

The object of this thesis is twofold:

1. to analyze and acquire a general and precise view of the existing numerical and enumerative algorithms for MPECs by conducting a comprehensive literature review through the different applications in various fields (control, operations research, game theory, supervised learning) and the recent advances in related domains (bilevel and disjunctive programming, interior and sequential-LP/QP methods, etc.);

2. to investigate the nonlinear case (i.e., taking nonlinear $g$) and develop novel solutions for classes of challenging nonconvex optimization problems – including the Standard Fractional Quadratic Program [2] and certain integer-nonlinear bilevel programs [1] with suitable duality properties – based on solving their MPEC reformulations with a combination

of techniques from different paradigms, such as smooth and discrete optimization or machine learning.

## Direction and context

The PhD thesis will be realized in the Centre de Mathématiques Appliquées of Mines Paris - PSL, located at Sophia-Antipolis, France, under the supervision of:

- Valentina Sessa (valentina.sessa@minesparis.psl.eu) https://sites.google.com/site/sessavalentina/home
- Sophie Demassey (sophie.demassey@minesparis.psl.eu) https://sofdem.github.io/

## Profile of the candidate

### Essential

- MSc degree (or equivalent, giving access to doctoral studies) in Mathematics, Systems and Control, Computer Science, or a related field.
- Knowledge of continuous optimization.
- Coding skills in Matlab and/or Python and/or Julia.
- Strong attitude towards problem-solving and experimental activity.
- Good communication skills in written and spoken English.

### Preferred

- Knowledge of discrete optimization.

## How to apply

The complete application consists of the following documents to be sent as a single PDF file to the email addresses given above (**deadline: May 30, 2024**):

- CV
- one-page cover letter (clearly indicating available start date as well as relevant qualifications, experience, and motivation)
- University certificates and transcripts with marks/grades obtained in all courses (BSc and MSc degrees)
- up to two letters of recommendation.

All documents should be in French or English.

## References


[1] S. Demassey, V. Sessa, A. Tavakoli, Alternating direction method and deep learning for discrete control with storage, ISCO (2024) to appear in LNCS.

[2] J. Júdice, V. Sessa, M. Fukushima, Solution of Fractional Quadratic Programs on the Simplex and Application to the Eigenvalue Complementarity Problem, Journal of Optimization Theory and Applications, 193 (2022) 545–573.

[3] J. Júdice, V. Sessa, M. Fukushima: A two-phase sequential algorithm for global optimization of the standard quadratic programming problem (under review).

[4] Z.-Q. Luo, J.-S Pang, D. Ralph: Mathematical Programs with Equilibrium Constraints. Cambridge University Press (2011).

[5] V. Sessa, L. Iannelli, F. Vasca, A Complementarity Model for Closed-Loop Power Converters, IEEE Transactions on Power Electronics, 29 (2014) 6821-6835.

[6] H. Van Stackelberg: The Theory of Market Economy, Oxford University Press, Oxford (1952).