Lagrangean relaxation-based lower bounds for the RCPSP

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1 Introduction

An instance of the Resource Constrained Project Scheduling Problem (RCPSP) is given by m renewable resources with limited capacities $R_1, \ldots, R_m$ and a project of n activities with simple precedence constraints (we note $(i,j) \in E$ if activity j has to start after the end of activity i). Each activity i must be processed during p_i uninterrupted time units and requires a constant amount r_{ik} of each resource k. Two dummy activities 0 and $n+1$ represent the beginning and the end of the project, respectively. A feasible schedule S (defined by the starting times $S_1, \ldots, S_n$ of the activities) meets altogether precedence constraints ($S_j \geq S_i + p_i$ if $(i,j) \in E$) and resource constraints (at any time t, the total amount of a resource k required by all the activities in process at t can not exceed R_k). The objective is then to find a feasible schedule S whose total completion time or makespan S_{n+1} is minimal.

The RCPSP and its variants have numerous applications in practice and contain many well-known difficult problems as special cases, like *e.g.* shop scheduling. Hence, RCPSP is very attractive for researchers and its intractibility has led many of them to design elaborated resolution methods. For instance, the currently strongest lower bounds of the minimal makespan (Brucker and Knust (2000), Demassey et al. (2003), Baptiste and Demassey (2003)) for the standard PSPLIB benchmark instances are computed in a destructive way by hybrid constraint programming-linear programming methods.

The approach presented in this paper is related to these methods: constraint propagation techniques are ran in a preprocessing phase of an integer linear formulation to compute and tighten time windows for activities ($S_i \in [ES_i, LS_i]$). A lower bound is then directly derived, in a constructive way, by solving a relaxation of the linear program, or, in a destructive way, by finding the greatest value T for which the constraint and the linear relaxations prove that no schedule with makespan lower than T exists.

Within the linear programming phase, we focus on the formulation given by Mingozzi et al. (1998) and on its preemptive relaxation lately improved by Brucker and Knust (2000). Both models contain an exponential number of binary variables, each one corresponding to a feasible set, *i.e.* a subset of activities that may be processed simultaneously without violating neither precedence nor resource constraints. The well-known $LB3$ lower bound is obtained by Mingozzi et al. (1998) with an heuristic solving a set-packing problem equivalent to the dual program of the preemptive relaxation. Alternatively, Brucker and Knust (2000) solve the preemptive relaxation by column generation. In this paper, we present a third way to tackle the size of this program by means of lagrangean relaxation. The proposed decomposition exhibit easily-solved substructures as small-sized multi-knapsack problems. The same technique is also applied to the initial complete model, then exhibiting, in addition within each lagrangean subproblem, one minimum cut problem, solvable in polynomial time (Möhring et al. (2003)).

2 Integer Linear Model with Feasible Sets

A feasible set (which can be the empty set) is denoted by F_l with $l \in \mathcal{F}$ and one containing a given activity i is indexed by $l \in \mathcal{F}_i$. A feasible schedule with makespan lower than T is defined by a sequence $(l_0, \ldots, l_{T-1})$ where F_{l_t} is the set (actually a feasible set) of activities in process between times t and $t+1$. With this observation, Mingozzi et al. (1998) propose the following integer linear model for the RCPSP:

$$
\begin{array}{llll}
(P) & \min \sum_{t=0}^{T} t y_{(n+1)t} & & (1)\\
& s.t.: \ \sum_{t=0}^{T} y_{it} = 1 & \forall\ i \in \{0, \ldots, n+1\} & (2)\\
& \sum_{t=0}^{T} t(y_{jt} - y_{it}) \geq p_i & \forall\ (i,j) \in E & (3)\\
& \sum_{l \in \mathcal{F}_i} \sum_{t=0}^{T-1} x_{lt} = p_i & \forall\ i \in \{1, \ldots, n\} & (4)\\
& \sum_{l \in \mathcal{F}} x_{lt} \leq 1 & \forall\ t \in \{0, \ldots, T-1\} & (5)\\
& y_{it} \geq \sum_{l \in \mathcal{F}_i} x_{lt} - \sum_{l \in \mathcal{F}_i} x_{lt-1} & \forall\ t \in \{0, \ldots, T-1\}, \forall\ i \in \{1, \ldots, n\} & (6)\\
& x_{lt} \in \{0,1\},\ x_{l(-1)} = 0 & \forall\ l \in \mathcal{F}, \forall\ t \in \{0, \ldots, T-1\} & (7)\\
& y_{it} \in \{0,1\} & \forall\ i \in \{0, \ldots, n+1\}, \forall\ t \in \{0, \ldots, T\} & (8)
\end{array}
$$

This program contains two kinds of binary variables: $x_{lt} = 1$ if l is the feasible set which is "active" at time t, and $y_{it} = 1$ if activity i starts at time t.

We propose to dualize both constraints (6), with multipliers $\lambda_{it} \in \mathbb{R}_+$, $i \in \{1, \ldots, n\}$ and $t \in \{0, \ldots, T\}$, and constraints (4), with multipliers $\mu_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$. For a given value (λ, μ) of the multipliers, the lagrangean subproblem can then be splitted into $T+1$ parts according to the independency of the variables: one program $(SP^t_{\lambda\mu})$ for each $t \in \{0, \ldots, T-1\}$ with the set of variables $\{x_{lt} \mid l \in \mathcal{F}\}$ and the corresponding constraints among (5), and one program (SP_λ) with variables $\{y_{it} \mid i \in \{0, \ldots, n+1\}, t \in \{0, \ldots, T\}\}$ and constraints (2) and (3).

Formally, the lagrangean dual of (P) may be written:

$$
(DLP) : \bar{z} = \max_{\lambda, \mu} \sum_{i=1}^{n} \mu_i p_i + \sum_{t=0}^{T-1} \phi^t_{\lambda\mu} + \psi_\lambda,
$$

where $\phi^t_{\lambda\mu}$ and ψ_λ are respectively the optimal values of the binary integer linear programs $(SP^t_{\lambda\mu})$ and (SP_λ).

Each subproblem $(SP^t_{\lambda\mu})$ consists in making active at time t a feasible set l_t which minimizes a certain cost $\sum_{i \in F_{l_t}} \nu_{it}$ where ν depends on λ and μ. It can be then reformulated as a quickly-solved multi-knapsack problem: l_t is such an optimal set of activities that can be executed altogether at time t, according to the resource and precedence constraints, but also according to the time windows computed within the preprocessing phase.

On other hand, the subproblem (SP_λ) is a project scheduling problem without resource constraints but with time-depending costs. This problem has recently been studied by Möhring et al. (2003) for an other lagrangean relaxation approach for the RCPSP. They show that this problem is equivalent to find a cut of minimal capacity in a digraph and propose then a polynomial time algorithm to solve it. Here again, constraint programming deductions can be useful to reduce the size of the digraph and then to speed up the resolution.

3 The Preemptive Relaxation

Mingozzi et al. (1998) consider also a relaxation of program (P) allowing preemption of activities and partly removing precedence constraints. Powerful lower bounds are then derived by Mingozzi et al. (1998) and by Brucker and Knust (2000) who enhance the formulation by taking into account the time windows computed for a given value T: time horizon $[0, T]$ is partitioned with the different values of ES_i and $LS_i + p_i$ into σ subintervals. For each period $s \in \{1, \dots, \sigma\}$, the feasible sets $\{F_l \mid l \in \mathcal{F}^s\}$, of activities that can be in process during period s are treated separately. Furthermore, they use a destructive approach and consider for different values of T a decision variant of the problem. We propose a quasi-dual approach of the column generation method of Brucker and Knust (2000) in using lagrangean relaxation on a slightly different linear program:

$$
\begin{array}{llll}
(PP) & \min \sum_{i=1}^{n} e_i & & (9) \\
& s.t.: \ \sum_{l \in \mathcal{F}^s} z_{ls} \le \delta_s & \forall\ s \in \{1, \dots, \sigma\} & (10) \\
& \sum_{s=1}^{\sigma} \sum_{l \in \mathcal{F}_i \cap \mathcal{F}^s} z_{ls} + e_i \ge p_i & \forall\ i \in \{1, \dots, n\} & (11) \\
& z_{ls} \ge 0 & \forall\ s \in \{1, \dots, \sigma\},\ l \in \mathcal{F}^s & (12) \\
& e_i \ge 0 & \forall\ i \in \{1, \dots, n\} & (13)
\end{array}
$$

Here, integer variable z_{ls} represents the total time during which the feasible set is active within interval s, and δ_s the length of s. No feasible schedule exists for the given time windows if the optimal value of program (PP) is strictly greater than 0.

By dualizing constraints (11), the resolution of the lagrangean subproblem amounts again to solve, for each period $s \in \{1, \dots, \sigma\}$, a multi-knapsack program and to compute exactly one feasible set active within s.

4 Experimental Results

Both constructive and destructive lower bounds may be derived from either the complete or the preemptive linear program, just taking the suitable objective function. We experiment the computation of the destructive lower bound with lagrangean relaxation for the preemptive model on the PSPLIB benchmark instances with 30 and 60 instances (Kolisch et al. (1997)).

The same constraint programming algorithm as described in Demassey et al. (2003) is used in the preprocessing phase. It includes the deduction techniques also used by Brucker and Knust (2000) such as disjunctive edge-finding or symmetric triple rules. It also includes a global technique of shaving. We use Ilog Cplex for the resolution of the multi-knapsack programs and solve the lagrangean dual iteratively with a subgradient algorithm. In the destructive process, values of horizon T are taken by dichotomy on an interval $[0, UB]$ where UB is the upper bound used by Brucker and Knust in the computational experiments of their bound (`BK`).

For the 480 instances with 30 activities, the quality of our bound is proved since it is strictly better than `BK` in terms of deviation from the optimum (0.8% vs. 1.5%) or in the number of instances for which optimum is reached (392 vs. 318) and despite of a processing time overcost, with an average CPU time of 14.3 seconds on a Pentium III 833 MHz versus 0.4 seconds for `BK` computed on a Sun Ultra 2 station running at 167MHz.

Not all the 480 instances with 60 activities are solved to optimality to date. So we compare the bounds over the deviation from the best upper bound available (`UB`) at the PSPLIB library. Again our bound is actually more time consuming than `BK` (195 seconds in average vs. 5 seconds) but it remains also greater in average with a deviation from `UB` of 1.9% (against 2.0% for `BK`) and 360 optimum proofs (against 341). Our bound dominates the best known lower bound (also available at PSPLIB) for 43 instances among the 124 not yet solved and allows to close 10 new instances.

Additional work has now to be done for a better parametrization to speed up the resolution of the lagrangean dual but our first experimented bound is clearly competitive with the best lower bounds to date. Also, it is really promising and encourage us to test and comparate the other bounds relative to this one, by considering the lagrangean relaxation of the complete model or by means of a constructive approach.

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