MONOTROPIC BILEVEL PROGRAMMING: DUALITY IN HYDRAULIC NETWORK OPTIMIZATION

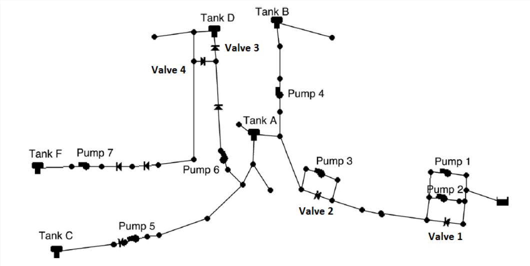
Sophie Demassey (Mines Paris – PSL)

GDR ROAD - GrT OR - 20/10/2023

How do hydraulic simulators work ?

How to use them in hydraulic network optimization ?

DRINKING WATER DISTRIBUTION NETWORK



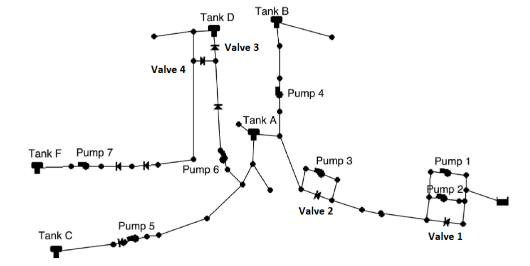
- simple directed graph $G = (J, A)$
- $a \in A$ = { pipes, pumps, valves }
- $j \in J$ = Service ∪ Reservoirs
- incidence matrix $E \in \{0, 1, -1\}^{A \times J}$:

$$a = (i, j) : E_{ai} = -1, E_{aj} = 1, E_{an} = 0$$

Hypothesis (for this exposé):

- no pressure-induced leakage, no aging
- fixed speed pumps (on/off), controlled gate valves (close/open)

DRINKING WATER DISTRIBUTION NETWORK



- flow q_a on arcs $a \in A$
- head h_j at nodes $j \in J$
- demand D_s at service nodes $s \in S$
- level/height H_r of reservoirs $r \in R$
- resistance ϕ_a on arcs $a \in A$

Network Analysis Problem: find (q_A, h_J) meeting (D_S, H_R, ϕ_A) ?

HYDRAULIC NETWORK ANALYSIS PROBLEM

$NAP(D_S, H_R, \phi_A) =$

$\{(q_A, h_J) \in \mathbb{R}^A \times \mathbb{R}^J,$		(flow, head)
$q_s = D_s$	$\forall s \in S,$	demand
$h_r = H_r$	$\forall r \in R,$	level
$v_a = \phi_a(q_a)$	$\forall a \in A\}$	resistance

where $q_j := \sum_a E_{aj} q_a$ — residual flow at node $j \in J$

$v_a := \sum_j -E_{aj} h_j$ — head loss on arc $a \in A$.

$$\implies \sum_a v_a q_a = -\sum_j h_j q_j$$

compute an element of NAP and check the bounds:

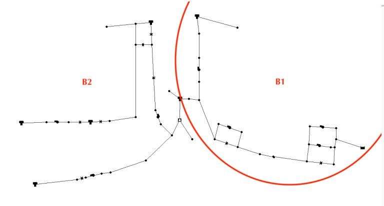
$NAP(D_S, H_R, \phi_A) =$

$$\begin{aligned}
&\{(q_A, h_S) \in \mathbb{R}^A \times \mathbb{R}^S, \\
&q_s = D_s && \forall\, s \in S, \\
&v_a = \phi_a(q_a) && \forall\, a \in A\}.
\end{aligned}$$

System of equations solved by the Newton-Raphson algorithm [Todini&Pilati 88, Salgado 89] example: EPANET

DECOMPOSITION OF NAP

$G = \cup_{b \in B}(J_b, A_b)$
graph partition along nodes in R



$NAP(D_S, H_R, \phi_A) =$

$$\bigcup_{b \in B} NAP(D_{S_b}, H_{R_b}, \phi_{A_b})$$

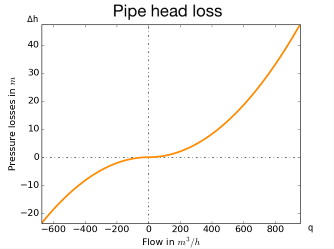
$$= \bigcup_{b \in B} \{(q_{A_b}, h_{S_b}) :$$

$$q_s = D_s \qquad \forall s \in S_b$$

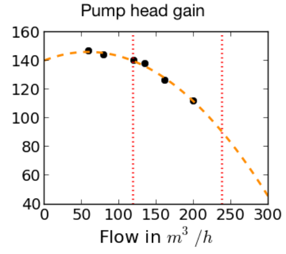
$$v_a = \phi_a(q_a) \qquad \forall a \in A_b\}$$

RESISTANCE

pipes: frictions (Darcy-Weisbach/Swamee-Jain)



pumps: discharge pressure (quadratic fit)



- quadratic approximation $\phi_a(q) = \alpha_a q|q| + \beta_a q + \gamma_a$ with $\alpha_a > 0$: continuous, strictly increasing, bijective on $\mathbb{R}$
- integral $f_a(q) = \int_0^q \phi_a(x)dx$ is smooth, strictly convex, and coercive
- inverse ϕ_a^{-1} has the same property

second option for solving NAP:

primal/dual reformulation

PRIMAL REFORMULATION OF NAP

(q_A, h_S) in

$$NAP: \quad q_s = D_s \forall s \in S,\ v_a = \phi_a(q_a) \forall a \in A$$

if and only if

q_A solves

$$P_{NAP}: \min_{q_A} \sum_{a \in A} f_a(q_a) + H_R^\top q_R \ : \ q_s = D_s \ \forall s \in S$$

with $f_a = \int \phi_a$ strictly convex, then solution is unique

Proof: NAP are the stationary points $\nabla L = 0$ of the lagrangian function:

$$L(q_A, h_S) = \sum_{a \in A} (f_a(q_a) - v_a q_a) - h_S^\top D_S.$$

DUAL REFORMULATION OF NAP

$$L(q_A, h_S) = \sum_{a\in A}\left(f_a(q_a) - v_a q_a\right) - h_S^\top D_S$$

Strong duality holds: $P \equiv D : \max_{h_S} \min_{q_A} L(q_A, h_S) = \max_{h_S} L((\phi_a^{-1}(v_a))_{a\in A}, h_S)$.

(q_A, h_S) in

$$NAP : \quad q_s = D_s \forall\, s \in S,\; v_a = \phi_a(q_a) \forall\, a \in A$$

if and only if

h_S solves

$$D_{NAP} : \min_{h_S} \sum_{a\in A} f_a^*(v_a) + D_S^\top h_S$$

with $f_a^*(v) = \max_q (vq - f_a(q)) = -f_a(\phi_a^{-1}(v)) + v\phi_a^{-1}(v)$ convex conjugate of f_a.

q_A minimizes F in P_{NAP} and h_S maximizes F^* in D_{NAP} then $F(q_A) \leq F^*(h_S)$

$NAP = SD_{NAP}$

$$SD_{NAP} = \{(q_A, h_S) \in \mathbb{R}^A \times \mathbb{R}^S,\ q_s = D_s \forall s \in S, \sum_{a \in A} \left(f_a(q_a) + f_a^*(v_a)\right) + H_R^\top q_R + D_S^\top h_S \leq 0\} \tag{SD}$$

with $f_a \in \int \phi_a$, $f_a(0) = 0$ and $f_a^* \in \int \phi_a^{-1}$, $f_a^*(0) = -f_a(\phi_a^{-1}(0))$.

STRONG DUALITY REFORMULATION OF NAP

$NAP = SD_{NAP}$

$$SD_{NAP} = \{(q_A, h_S) \in \mathbb{R}^A \times \mathbb{R}^S,\ q_s = D_s \forall s \in S, \\ \sum_{a \in A} \left(f_a(q_a) + f_a^*(v_a) \right) + H_R^\top q_R + D_S^\top h_S \leq 0\} \qquad (SD)$$

with $f_a \in \int \phi_a$, $f_a(0) = 0$ and $f_a^* \in \int \phi_a^{-1}$, $f_a^*(0) = -f_a(\phi_a^{-1}(0))$.

- (SD) integrates and aggregates the flow-potential equations:

$$(SD) \iff \sum_a f_a(q_a) + f_a^*(v_a) - q_a v_a = 0$$

$$\iff f_a(q_a) = f_a(\phi_a^{-1}(v_a)) + f_a'(\phi_a^{-1}(v_a))(q_a - \phi_a^{-1}(v_a))\ \forall a$$

$$\iff \phi_a^{-1}(v_a) = q_a\ \forall a.$$

- SD_{NAP} is an exact aggregate reformulation of NAP
- P_{NAP} and D_{NAP} are conjugate convex nonlinear programs
- called *content* and *co-content* models in [Collins 1978]
- or *distribution* and *differential* problems in [Rockafellar 1988]
- generalization: nonlinear flow networks and monotropic programs [Rockafellar 1988]

MONOTROPIC PROGRAMMING [ROCKAFELLAR, 1988]

additive convex objective over linear constraints

$$P : \min_{x \in \mathbb{R}^J} \sum_{j \in J} f_j(x_j)$$

$$s.t. \sum_{j \in J} E_{ij} x_j = d_i \qquad \forall i \in I$$

f_j closed proper convex on $\mathbb{R}$ = lower semi-continuous (poss. nonsmooth)

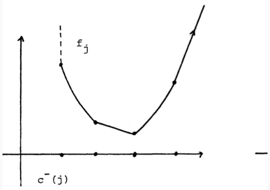


Figure 8.24

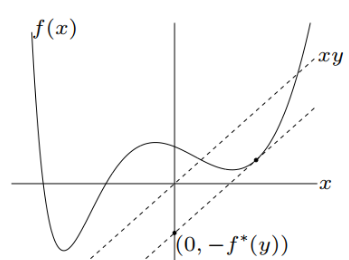
- monotropic = "one-dimension convexity" (extended to finite-dimension in [Bertsekas 2008])
- a class of convex programs behaving like linear programs:
    - combinatorial properties: finite set of descent directions (*elementary vectors*)
    - duality properties: strong duality, explicit symmetric dual

MONOTROPIC PROGRAMMING: (FENCHEL) DUALITY

$f^* : v \in \mathbb{R} \mapsto \sup_x (xv - f(x))$ convex conjugate of f (Legendre-Fenchel transformation)

$$(P) : \min_{x \in \mathbb{R}^J} \sum_{j \in J} f_j(x_j) \quad s.t. \sum_{j \in J} E_{ij} x_j = d_i \quad \forall i \in I$$

$$(D) : \min_{u \in \mathbb{R}^I} \sum_{i \in I} d_i u_i + \sum_{j \in J} f_j^*(v_j) \quad s.t.\ v_j := \sum_{i \in I} -E_{ij} u_i \quad \forall j \in J$$



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- *conjugate* f_j^* is convex lower semi-continuous: D is monotropic
- *biconjugate* $f_j = f_j^{**}$ (as f_j convex l.s.c.): dual(dual)=primal
- *Fenchel inequality*: $f_j(x_j) + f_j^*(v_j) \geq x_j v_j$ and equality holds iff $v_j \in \partial f_j(x_j)$
- strong duality and KKT conditions for $(x; u, v)$ a feasible primal-dual pair:
  $0 = \sum_j \left(f_j(x_j) + f_j^*(v_j)\right) + \sum_i d_i u_i = \sum_j \left(f_j(x_j) + f_j^*(v_j) - x_j v_j\right) \iff v_j \in \partial f_j(x_j) \forall j$

MONOTROPIC PROGRAMMING: EQUIVALENT CONDITIONS (FINITE OPTIMUM)

primal: x solves

$$(P) : \min_{x} \sum_{j} f_j(x_j)$$

$$s.t. \sum_{j} E_{ij} x_j = d_i \qquad \forall i$$

dual: u solves

$$(D) : \min_{u} \sum_{i} d_i u_i + \sum_{j} f_j^*(v_j)$$

$$s.t.\ v_j := \sum_{i} -E_{ij} u_i \qquad \forall j$$

equilibrium (KKT): (x, u) solves

$$(Eq) : \sum_{j} E_{ij} x_j = d_i \qquad \forall i$$

$$v_j := \sum_{i} -E_{ij} u_i \in \partial f_j(x_j) \qquad \forall j$$

strong duality: (x, u) solves

$$(SD) : \sum_{j} E_{ij} x_j = d_i \qquad \forall i$$

$$\sum_{j} \left(f_j(x_j) + f_j^*(v_j) \right) + \sum_{i} d_i u_i \leq 0.$$

MONOTROPIC PROGRAMMING: APPLICATIONS

1. f_j piecewise linear/quad-convex

no need to linearize to dualize

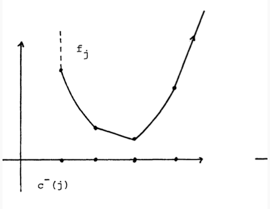


Figure 8.24

$$(P): \min_x \sum_j f_j(x_j)$$

$$s.t. \sum_j E_{ij} x_j = d_i \qquad \forall\, i$$

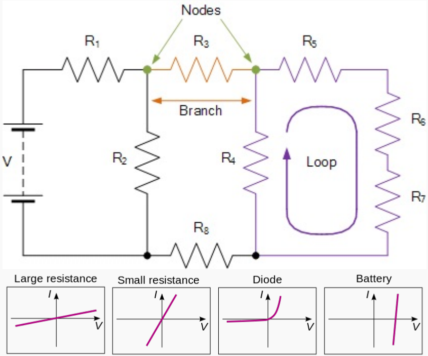
2. potential-flow network

$$(Eq): \sum_j E_{ji} x_j = d_i \qquad \forall\, i$$

$$v_j = \sum_i -E_{ji} u_i \in \partial f_j(x_j) \qquad \forall\, j$$

- E incidence matrix of graph G(I,J)
- x arc flows, u node potentials
- ∂f arc resistance/conductivity

- equilibrium problem = NAP in hydraulic
- model for many other physical networks (newtonian): electricity, gas, heat, telecommunications, transportation, vascular, elastic/spring



- ex: electric circuit
- A: conductors (resistors, batteries,...) with linear resistance $r = v/x$ (Ohm's law)
- x current, v voltage
- flow conservation = Kirchhoff's current law

EX: EQUILIBRIUM WITH LINEAR RESISTANCE

$$\phi(x) = rx$$

- laws of Ohm (electric), Fourier (thermal), Poiseulle (viscous fluids)
- equilibrium solution minimizes energy dissipation:

$$(P) : \min_{x, Ex=d} \sum_j f_j(x_j) = \frac{r_j}{2} x_j^2 \text{ with } f_j = \int \phi_j.$$

applications to hydraulic network optimization

design

- gravity-fed network
- static demand
- installation costs
- alternative arcs

operation

- pressurized network
- dynamic demand
- energy costs
- controllable arcs

bilevel structure

1. select a subset of arcs $A' \subseteq A$
2. NAP: find an equilibrium on A' satisfying the demand

selection step: incomplete (metaheuristics) or implicit search (math prog)

DESIGN: PIPE SIZING (STATIC)

- a graph $G = (J, A \times K)$ with replicated arcs (possible pipe dimensions)
- arc status $x_{ak} \in \{0,1\}$: pipe of type k selected on arc a

$$\min_{x,q,h} \sum_a \sum_k c_{ak} x_{ak}$$

$$s.t. x_{ak} = 0 \implies q_{ak} = v_{ak} = 0 \qquad \forall a \in A, k \in K$$

$$\sum_k x_{ak} = 1, h_i - h_j = \sum_k v_{ak} \qquad \forall a = (i,j) \in A$$

$$(q_{AK}, h_S) \in NAP(D_S, H_R, \phi_{AK(x)}).$$

Nonconvex MINLP formulation

DESIGN: PIPE SIZING (STATIC)

- a graph $G = (J, A \times K)$ with replicated arcs (possible pipe dimensions)
- arc status $x_{ak} \in \{0, 1\}$: pipe of type k selected on arc a

$$
\begin{aligned}
\min_{x,q,h} \sum_a \sum_k c_{ak} x_{ak} & \\
s.t. x_{ak} = 0 \implies q_{ak} = v_{ak} = 0 & \qquad \forall a \in A, k \in K \\
\sum_k x_{ak} = 1, h_i - h_j = \sum_k v_{ak} & \qquad \forall a = (i,j) \in A \\
\sum_{ak} E_{as} q_{ak} = D_s & \qquad \forall s \in S \\
\sum_{ak} \left(f_{ak}(q_{ak}) + f^*_{ak}(v_{ak}) \right) + H_R^\top q_R + D_S^\top h_S \leq 0 & \qquad (SD)
\end{aligned}
$$

Exact convex MINLP reformulation [TASSEF 2020], still non-polynomial

OPERATION: PUMP SCHEDULING (DYNAMIC + STORAGE)

- a dynamic graph $G = (J \times T, A \times T)$ and dynamic tariff c on discrete horizon T
- arc status $x_{at} \in \{0,1\}$: arc a active at time t
- variable tank level H_{rt} depends on $q_{r(t-1)}$

$$
\begin{aligned}
\min \sum_a \sum_t c^0_{at} x_{at} + c^1_{at} q_{at} & \\
s.t. (q_{At}, h_{St}) \in NAP(D_{St}, H_{Rt}, \phi_{A(x_t)}) & \quad \forall t \in T \\
x_{at} = 0 \implies q_{at} = 0 & \quad \forall a \in A, t \in T \\
H_{R(t+1)} = H_{Rt} + s_R^\top q_{Rt} & \quad \forall t \in T \\
\underline{H}_{Rt} \leq H_{Rt} \leq \overline{H}_{Rt} & \quad \forall t \in T.
\end{aligned}
$$

OPERATION: STRONG DUALITY REFORMULATION

strong duality constraints are not convex

$$\sum_{a \in A} \left(f_a(q_{at}) + f_a^*(v'_{at}) \right) + H_{Rt}^\top q_{Rt} + D_{St}^\top h_{St} \leq 0 \quad \forall t$$

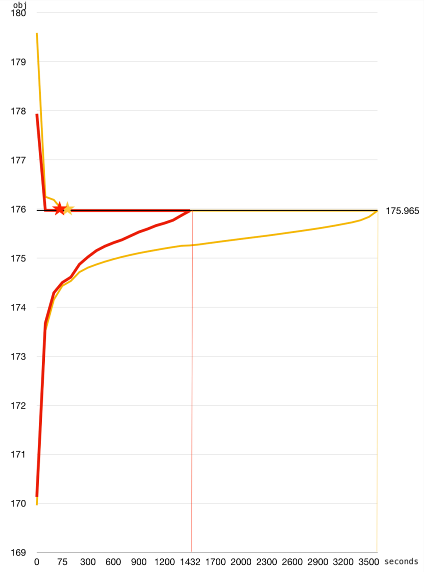
with $x_{at} = 1 \implies v'_{at} = v_{at}$ and $x_{at} = 0 \implies v'_{at} = (f_a^*)^{-1}(0)$

Option 1: relax and convexify

- $f_a(q_{at}) + f_a^*(v'_{at})$ is convex $\implies$ linearize at trial points
- bad news: a loose relaxation of the bilinear terms may *absorb* the duality gap
- good news: tank capacities provide exogenous bounds on H_{Rt}, $H_{R(t+1)}$ and q_{Rt} to tighten McCormick's relaxation

OPERATION - OPTION 1: CUT GENERATION

- Branch-and-Check [Bonvin, Demassey, Lodi 2020]
- evolution of the primal/dual bounds
- with or without duality cuts



OPERATION - OPTION 2: VARIABLE SPLITTING

$$\min_{x,q,h,H} \sum_a \sum_t (c^0_{at} x_{at} + c^1_{at} q_{at})$$

$$s.t. (q_{At}, h_{St}) \in NAP(D_{St}, H_{Rt}, \phi_{A(x_t)}) \qquad \forall t \in T$$

$$H_{R(t+1)} = H_{Rt} + s_R^\top q_{Rt} \qquad \forall t \in T$$

$$\underline{H}_{Rt} \leq H_{Rt} \leq \overline{H}_{Rt} \qquad \forall t \in T.$$

- complexity comes less from the nonconvex constraints $v_a = \phi_a(q_a)$, than from the temporal inter-dependency $q_t = F(x_t, H_t)$, and $H_{t+1} = G(q_t)$
- still hard when dualizing the time-coupling constraints as H remains variable
- fixing H allows to decompose the problem temporally and spatially, but we loose convergence

OPERATION - OPTION 2A: PENALIZE STORAGE (AMIR'S WORK)

$$\min_{x,q,h,H} \sum_a \sum_t (c^0_{at} x_{at} + c^1_{at} q_{at}) + \sum_r \sum_t \mu_{rt} |H_{r(t+1)} - (H_{rt} + s_r q_{rt})|$$

$$s.t.\ (q_{A_b t}, h_{S_b t}) \in NAP_b(D_{S_b t}, H_{R_b t}, \phi_{A_b(x_t)}) \qquad \forall t \in T, b \in B$$

$$\underline{H}_{Rt} \leq H_{Rt} \leq \overline{H}_{Rt} \qquad \forall t \in T.$$

$(P1)$: fix H, enumerate x, get q $\quad$ $(P2)$: fix q, relax NAP, get H $\quad$ 3: update μ

- $(P1)$ becomes decomposable both in time and space, thus enumerable
- not full split: then relax NAP in $(P2)$
- initial H obtained from a deep learning model

OPERATION - OPTION 2B: DUALIZE STORAGE + STRONG DUALITY

$$\min_{x,q,h,H} \sum_a \sum_t (c^0_{at} x_{at} + c^1_{at} q_{at}) + \sum_r \sum_t \mu_{rt}(H_{r(t+1)} - H_{rt} - s_r q_{rt}) + \sum_t \lambda_t SD_t(q,h,H)$$

$$s.t.\ q_{St} = D_{St} \qquad \forall t \in T$$

$$\underline{H}_{Rt} \leq H_{Rt} \leq \overline{H}_{Rt} \qquad \forall t \in T.$$

$(P1)$: fix H, enumerate x, get q $(P2)$: fix q, get H 3: update μ

- full split
- primal and dual objective function F and F^* now appear in the objective:

$$SD_t(q,h,H) = F(q) + F^*(h,H) = (\sum_{a \in A} f_a(q_{at}) + H^\top_{Rt} q_{Rt}) + (\sum_{a \in A} f^*_a(v'_{at}) + D^\top_{St} h_{St})$$

OPERATION - OPTION 2B: DUALIZE (STORAGE + SD) AND ADM (CONT.)

$$(P2) : \sum_t \sum_r \min_{H \in [\underline{H}, \overline{H}]} l(H) + F^*(H).$$

$(P2)$ is computed by minimizing univariate convex functions over intervals

$$(P1) : \sum_t \sum_b \min_x l(x) + F(\tilde{q}(x)) + F^*(\tilde{h}(x))$$

$(P1)$ is separable in time and space and results in solving two perturbed equilibrium problems for each configuration x

costs and penalties are reported to the lower NAP level.

CONCLUSION

- nonconvex resistance constraints are not that hard in hydraulic network optimization
- but dynamic storage management in pump scheduling is hard
- problems have a bilevel structure with NAP at the inner level
- ways to exploit NAP (monotropic) duality
- ways to exploit NAP (monotropic) variational properties ?

REFERENCES

- our presentations and papers on the pump scheduling problem are available on https://sofdem.github.io/
- code and benchmarks available on: https://github.com/sofdem/gopslpnlpbb