On Computational Evaluation of Lower Bounds for the Fractional Quadratic Program over the Standard Simplex

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Abstract

This work evaluates techniques for computing a lower bound for the Standard Fractional Quadratic Program (StFQP). These results may be useful to be integrated into algorithms to globally solve this type of problem. Moreover, we implement computational experiments to assess the trade-off between cost and tightness of the computed lower bounds.

Keywords: Fractional quadratic programming, global optimization.

1. Introduction

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ be real symmetric matrices, with $\mathbf{B}$ positive-definite (PD), and let the *standard simplex* be denoted by $\Delta \coloneqq \left\{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{e}^\top \mathbf{x} = 1\right\}$, where $\mathbf{e} = (1, \ldots, 1)^\top \in \mathbb{R}^n$. With this notation in place, we define the *Standard Fractional Quadratic Program* (StFQP) as follows

$$\min_{\mathbf{x}\in\Delta} \lambda(\mathbf{x}) \coloneqq \frac{\mathbf{x}^\top \mathbf{A}\mathbf{x}}{\mathbf{x}^\top \mathbf{B}\mathbf{x}}. \tag{1}$$

Since $\mathbf{B}$ is PD, the function λ is well-defined over the feasible set Δ. Problem (1) can model a wide range of applications in economics, finance, communication, and engineering [9]. In this case, the aim is to optimize the

performance of a given system expressed as the ratio of two functions to accurately represent the balance between two aspects of the system, such as production cost versus production time, return versus risk of an investment.

Contrary to the case where the numerator is a linear function and the denominator is concave, a stationary point (SP) of StFQP (1) is not necessarily a global minimum because the objective function is not quasi-convex. In this case, computing a global solution is known to be NP-hard [1].

The computation of a stationary point for StFQP can be implemented by exploiting its relation with the class of the symmetric *Eigenvalue Complementarity Problem* (EiCP) [9]. In fact, by applying the Karush–Kuhn–Tucker (KKT) conditions to (1), one shows that any SP $\bar{\mathbf{x}}$ with associated value $\bar{\lambda} = \lambda(\bar{\mathbf{x}})$ must satisfy the following system

$$\mathbf{x} \geq \mathbf{0}, \quad \mathbf{e}^\top \mathbf{x} = 1, \quad \mathbf{A}\mathbf{x} - \lambda \mathbf{B}\mathbf{x} \geq \mathbf{0}, \quad \mathbf{x}^\top (\mathbf{A}\mathbf{x} - \lambda \mathbf{B}\mathbf{x}) = 0, \qquad (2)$$

so that $(\bar{\lambda}, \bar{\mathbf{x}})$ is a solution of EiCP$(\mathbf{A}, \mathbf{B})$ defined in (2). This problem was introduced by Seeger [13] in the special case where $\mathbf{B}$ is the identity matrix, and later extended to any PD matrix $\mathbf{B}$ by Queiroz et al. [12]. Modern state-of-the-art algorithms for symmetric EiCPs (including ADMM [8], SPL [6], DC-programming [10], spectral BAS [2], etc.) can compute these complementarity eigenpairs from (2) and generate SPs of (1). By (2), the global minimizer of StFQP is the eigenvector associated with the *smallest complementary eigenvalue* $\lambda_{\min}$ of the pair $(\mathbf{A}, \mathbf{B})$. Then, thanks to the combinatorial nature of the EiCP, a procedure able to compute all complementarity eigenvalues could be applied and $\lambda_{\min}$ can be obtained after sorting these values. However, it is important to mention that the number of complementary eigenvalues grows exponentially with problem dimension, and enumerating all is only possible for small instances [5].

Other effective methods to calculate SPs of (1) include a sequential algorithm by Júdice et al. [9], which applies a efficient implementation of the Dinkelbach's method to the linear quadratic fractional problem obtained linearizing the numerator in (1). Moreover, Boţ et al. [4] propose an extrapolated proximal subgradient algorithm, which computes SPs of more general nonconvex and nonsmooth fractional programs.

2. On lower bounds for StFQP

In this work, we are interested in evaluating lower bounds for StFQP, that is, we search for ℓ which satisfies the following

$$\ell \leq \lambda(\mathbf{x}^*), \tag{3}$$

where $\lambda(\mathbf{x}^*)$ is the global optimal value of (1).

Computing a tight lower bound is essential in many solvers using branch and bound techniques, for pruning the search tree and speeding convergence. However, excessively costly computations for obtaining a good ℓ can prevent these benefits, so the art lies in balancing bound-tightening effort against enumeration work. Inspired by the study of Bomze et al. [3] regarding lower bounding techniques for the Standard Quadratic Program, and motivated by the need for an efficient global solver for the fractional case, this work presents a systematic comparison of both emerging and classical bounding strategies, with the dual aim of deepening theoretical insights and guiding an improvement of optimization algorithms for (1).

Early on, Preisig [11] derived lower (and upper) bounds for (1) by optimizing the quadratic forms in the numerator and denominator independently, using their extreme eigenvalues to localize the quotient.

Through copositive optimization and its tight semidefinite programming (SDP) relaxations, Amaral et al. [1] show that the StFQP can be reformulated as a linear optimization problem over the *completely positive cone* $\mathcal{C}_n^*$. From that, they derive a hierarchy of SDP relaxations by replacing $\mathcal{C}_n^*$ with the *doubly nonnegative cone* $\mathcal{P}_n \cap \mathcal{N}_n \supset \mathcal{C}_n^*$. These bounds strictly dominate the usual convex-envelope approaches (RLT/LP), although they are computationally expensive for large instances.

For the case where $\mathbf{B}$ is the identity, Fernandes et al. [5] introduced algebraic procedures for finding bounds based on the localization set of the classical eigenvalues associated with the matrix $\mathbf{A}$. Recently, He et al. [7] extended this result, proposing two Gershgorin-type localization sets for EiCP with $\mathbf{B}$ being the identity. In this work, we generalize these bounds to the case of an arbitrary PD matrix $\mathbf{B}$.

We additionally carry out a comparison of the lower bound obtained by the methods discussed above from both theoretical and computational aspects. This analysis aims to determine the effectiveness of each method in providing accurate lower bounds while considering the computational resources they require.

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