LOCALIZATION OF COMPLEMENTARITY EIGENVALUES

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Abstract. Let $\mathbf{A},\mathbf{B}$ be symmetric $n\times n$ real matrices with $\mathbf{B}$ positive definite and strictly diagonally dominant. We derive two localization sets for the complementarity eigenvalues of $(\mathbf{A},\mathbf{B})$, the tightest one assuming additionally that $\mathbf{A}$ is copositive. This extends He-Liu-Shen sets to the case where $\mathbf{B}$ is not the identity. Moreover, we compare the computable bounds obtained from these new sets with the extreme classical generalized eigenvalues.

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1. Introduction

Let $\mathbf{A},\mathbf{B}$ be symmetric $n\times n$ real matrices and $\mathbf{B}$ be positive definite. The symmetric *eigenvalue complementarity problem* (EiCP) is to find $\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ and $\lambda\in\mathbb{R}$ satisfying:

$$(\mathbf{A}-\lambda\mathbf{B})\mathbf{x}\geq\mathbf{0} \tag{1.1a}$$

$$\mathbf{x}\geq\mathbf{0} \tag{1.1b}$$

$$\mathbf{x}^\top(\mathbf{A}-\lambda\mathbf{B})\mathbf{x}=0 \tag{1.1c}$$

$$\mathbf{e}^\top\mathbf{x}=1, \tag{1.1d}$$

where $\mathbf{e}:=(1,\dots,1)^\top\in\mathbb{R}^n$ is the all-ones vector and the inequalities are meant to be componentwise. For a pair $(\mathbf{x},\lambda)$ satisfying (1.1), $\lambda\in\mathbb{R}$ is a *complementarity eigenvalue* and $\mathbf{x}\in\mathbb{R}^n$ is a *complementarity eigenvector*. The orthogonality condition (1.1c) together with the nonnegativity requirements (1.1a) and (1.1b) imply that $x_i=0$ or $w_i=0$ with $\mathbf{w}=(\mathbf{A}-\lambda\mathbf{B})\,\mathbf{x}$ for all $i=1,\dots,n$.

Seeger introduced this problem in [24] for the case when $\mathbf{B}$ is equal to the identity matrix. In this context, this problem is often referred to as the *Pareto eigenvalue problem* because the complementarity spectrum is then attached to a single matrix $\mathbf{A}$ under the nonnegativity cone (also known as Pareto cone). Queiroz et al. [22] subsequently extended the formulation to general symmetric pairs $(\mathbf{A},\mathbf{B})$, motivated by the stability analysis of mechanical systems with frictional contacts. The practical importance of EiCP is further underscored by applications in copositive matrix analysis [12], asymptotic studies of Fucik curves [13], graph theory [9, 25], and linear dynamical systems governed by complementarity conditions [24].

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Theoretical and numerical properties of the symmetric EiCP have been investigated in recent years, as well as different generalizations [4, 5, 7, 15, 16]. The numerical resolution of EiCP is often based on its reformulation as a more manageable problem, for which specialized algorithms can be designed. For instance, in [17], the EiCP is formulated as a difference-of-convex program over the standard simplex, then solved with an alternating direction method of multipliers, which globally converges to a solution of the symmetric EiCP. In [10], the symmetric EiCP is reformulated as finding a stationary point to the minimization of the sum of a convex function and a smooth but nonconvex function. An algorithm, that solves a sequence of convex quadratic problems obtained by linearizing the nonconvex term, is shown to converge to a solution of the EiCP under mild assumptions. A different reformulation, as a fractional program, is to minimize the ratio of two quadratic functions, namely the generalized Rayleigh quotient, over the standard simplex [18, 22]. In [19], a stationary point is computed by using an efficient implementation of the Dinkelbach's method [23] after linearizing the numerator of the objective function around a current iterate. In [2], the EiCP is formulated as an equivalent nonsmooth system of equations based on complementarity functions [6] and solved using a semi-smooth Newton method. In [1], the EiCP is formulated as a nonlinear system of equations, solved with two versions of the interior point method, which appear to be numerically more efficient than the smoothing method on medium-sized instances ($n \approx 100$), but not on smaller instances.

EiCP (1.1) extends the generalized eigenvalue problem [14], defined as $\mathbf{Ax} = \lambda \mathbf{Bx}$, that is (1.1a) satisfied at equality and dropping (1.1b) and (1.1c). Note that if $(\mathbf{x}, \lambda)$ solves (1.1) with $x_i > 0$, for all $i = 1, \ldots n$, the orthogonality constraint (1.1c) forces $\mathbf{Ax} - \lambda \mathbf{Bx} = \mathbf{0}$. More generally, if a subvector $\mathbf{x}_S > \mathbf{0}$ on an index set $S \subset \{1, \ldots, n\}$, then $(\mathbf{x}_S, \lambda)$ is a generalized eigenpair of the submatrices $(\mathbf{A}_{SS}, \mathbf{B}_{SS})$ [8]. This submatrix perspective suggests a constructive path: by scanning principal submatrices and computing classical generalized eigenpairs, one can enumerate complementarity eigenvalues [24]. This procedure is computationally viable only for very small instances ($n \leq 15$). Unlike the generalized eigenvalue problem, which has at most n solutions, the system (1.1) may exhibit exponentially many complementarity eigenpairs as n grows, at most $2^n - 1$, then exhaustive enumeration becomes prohibitive [8].

Recently, He et al. [11] resolved an open question of Seeger [24], regarding the location of the complementarity eigenvalues when $\mathbf{B}$ is the identity matrix. The authors derived computable localization sets, as intervals in $\mathbb{R}$, that depend only on row sums of $\mathbf{A}$, in a way related to the Gershgorin circle theorem [14] for ordinary eigenvalues. These results provide explicit algebraic bounds for λ.

In this paper, we address the same question for the symmetric EiCP (1.1) with $\mathbf{B}$ strictly diagonally dominant. It is worth mentioning that generalizing $\mathbf{B}$ from the identity matrix to any $\mathbf{B}$ positive definite and strictly diagonally dominant represents a contribution in this context. Although it may appear restrictive, the positive definiteness of $\mathbf{B}$ is commonly assumed in EiCPs to ensure the existence of a solution [8, 10, 17, 19, 22], and the diagonal dominance assumption is also standard and frequently employed in the localization theory of various eigenvalue classes via Gershgorin-type sets [20, 21, 26]. Extending the approach in [11], we present localization sets, based on either single rows or pairs of rows, under the extra assumption that $\mathbf{A}$ is copositive in the latter case. When $\mathbf{B}$ is the identity, our formulas reduce to those in [11], thereby recovering the Pareto setup. We also show that the two-row refinement provides tighter sets than the one-row analysis and that there is no dominance between the bounds provided by these localization sets and the extreme (largest and smallest) generalized eigenvalues.

This paper is organized as follows: first we introduce the necessary notation and preliminary concepts. In Section 2, we derive the one-row localization set K_1 and the two-row localization set K_2 under the additional hypothesis that $\mathbf{A}$ is copositive. We then prove in Section 3 that K_2 is contained in K_1. Section 4 extracts computable lower and upper bounds from these sets. Finally, Section 5 compares our localization sets with the standard generalized eigenvalue spectrum, and Sections 6 and 7 provide concluding remarks and discuss potential extensions.

Notation and Preliminaries

We indicate with $[n]$ the set of natural numbers between 1 and n, i.e., $[n] := \{1, ..., n\}$. Let $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{S}^n$ denote the n-dimensional Euclidean space, the nonnegative orthant, and the space of $n \times n$ real symmetric matrices, respectively.

For a vector $\mathbf{x} \in \mathbb{R}^n$, we denote its j-th component by x_j, $j \in [n]$. The element (i, j) of a matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ is denoted as m_{ij}, and $\mathbf{M}_i^\top$ represents the i-th row of $\mathbf{M}$.

The j-th standard basis vector in $\mathbb{R}^n$ is denoted by $\mathbf{e}_j$. We denote by $\mathbf{e} \in \mathbb{R}^n$ the n-dimensional vector of ones and by $\mathbf{E} = \mathbf{e}\mathbf{e}^\top \in \mathbb{S}^n$ the matrix with all elements being equal to one.

Definition 1.1 (Diagonal dominance)**.** Let $\mathbf{M} \in \mathbb{R}^{n\times n}$. We say that $\mathbf{M}$ is *diagonally dominant* (DD) if $|m_{ii}| \geq \sum_{j\neq i} |m_{ij}|$ for every $i \in [n]$. It is *strictly diagonally dominant* (SDD) if $|m_{ii}| > \sum_{j\neq i} |m_{ij}|$ for every $i \in [n]$.

Definition 1.2 (Positivity)**.** Let $\mathbf{A} \in \mathbb{S}^n$. We say that $\mathbf{A}$ is *positive semidefinite* (PSD) if $\mathbf{x}^\top\mathbf{A}\,\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. It is *positive definite* (PD) if $\mathbf{x}^\top\mathbf{A}\,\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$.

Definition 1.3 (Copositivity)**.** Let $\mathbf{A} \in \mathbb{S}^n$. We say that $\mathbf{A}$ is *copositive* if $\mathbf{x}^\top\mathbf{A}\,\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$. It is *strictly copositive* if $\mathbf{x}^\top\mathbf{A}\,\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} > \mathbf{0}$.

Clearly, any nonnegative or positive definite matrix is copositive. It is also known that symmetric SDD matrices with positive diagonal elements are PD (see Thm. 6.1.10 in [14]).

Property 1.4. EiCP$(\mathbf{A}, \mathbf{B})$ *has a solution* $(\mathbf{x}, \lambda)$ *if and only if* EiCP$(\mathbf{A} + \mu\mathbf{B}, \mathbf{B})$ *has a solution* $(\mathbf{x}, \lambda + \mu)$ *with* $\mu \geq 0$.

Note that since $\mathbf{B}$ is PD, then for some $\mu > 0$ the matrix $\mathbf{A} + \mu\mathbf{B}$ is also PD [14]. So, we can assume without loss of generality that $\mathbf{A}$ is a PD (or at least copositive) matrix in EiCP (1.1).

Property 1.5. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ *and* $\mathbf{B}$ *be positive definite. If* $\mathbf{A}$ *is a copositive matrix, then in any solution* $(\mathbf{x}, \lambda)$ *of the* EiCP$(\mathbf{A}, \mathbf{B})$*, the complementarity eigenvalue* λ *is nonnegative.*

A solution $(\mathbf{x}, \lambda)$ of the EiCP satisfies the orthogonality condition (1.1c), then $\mathbf{x} \geq \mathbf{0}$ and

$$\lambda = \frac{\mathbf{x}^\top\mathbf{A}\mathbf{x}}{\mathbf{x}^\top\mathbf{B}\mathbf{x}} \tag{1.2}$$

is the *generalized Rayleigh quotient* of $(\mathbf{A}, \mathbf{B})$ [22]. If $\mathbf{A}$ is copositive and $\mathbf{B}$ is PD, then by Definitions 1.2 and 1.3, we obtain the result in Property 1.5.

2. Localization sets for EiCP solutions

If $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices, the complementarity eigenvalues can be located easily. Indeed, they are the points a_{ii}/b_{ii}, $i \in [n]$, in the real space [24]. As done for Gershgorin circle theorem [14] for the classical eigenvalue problem in the complex space, we consider localizing the complementarity eigenvalues with respect to the points a_{ii}/b_{ii} when the matrices $\mathbf{A}$ and $\mathbf{B}$ are not diagonal. The first localization set examines how far the matrices are to be diagonal, by regarding the off-diagonal entries row-by-row.

Definition 2.1. For any matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, and row $i \in [n]$, the row sums of positive and negative off-diagonal entries and the associated diagonal shifts are denoted as follows:

$$r_i^+(\mathbf{M}) := \sum_{j\neq i} \max\{m_{ij}, 0\}, \quad r_i^-(\mathbf{M}) := -\sum_{j\neq i} \min\{m_{ij}, 0\}.$$

$$m_i^+ := m_{ii} + r_i^+(\mathbf{M}), \quad m_i^- := m_{ii} - r_i^-(\mathbf{M}).$$

Note that $m_i^+ > 0$ for matrices that are positive definite or strictly copositive. However, the sign of m_i^- is not determined a priori for these classes of matrices.

Theorem 2.2 (One-row localization)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}_n$, and assume that $\mathbf{B}$ is positive definite and strictly diagonally dominant. Let $\lambda \in \mathbb{R}$ be a complementarity eigenvalue of $(\mathbf{A}, \mathbf{B})$, then*

$$\lambda \in \bigcup_{i\in[n]} \left[\min\left(\frac{a_i^-}{b_i^-}, \frac{a_i^-}{b_i^+}\right), \max\left(\frac{a_i^+}{b_i^-}, \frac{a_i^+}{b_i^+}\right)\right] =: K_1.$$

To prove this theorem, we make use of the following result.

Lemma 2.3. *Let $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$ and any $p \in \arg\max_{i\in[n]} x_i$, then:*

(i) $m_p^- x_p \leq \mathbf{M}_p^\top \mathbf{x} \leq m_p^+ x_p$
(ii) *if $\mathbf{M}$ is strictly diagonally dominant and $m_{ii} \geq 0$, then $m_i^+ \geq m_i^- > 0$.*

Proof. Since $x_p \geq x_j \geq 0$, for $j \neq p$,

$$-r_p^-(\mathbf{M})x_p \leq \sum_{\substack{j\neq p\\ m_{pj}<0}} m_{pj}x_j \leq \sum_{j\neq p} m_{pj}x_j \leq \sum_{\substack{j\neq p\\ m_{pj}>0}} m_{pj}x_j \leq r_p^+(\mathbf{M})x_p.$$

Adding $m_{pp}x_p$ yields (i). If $\mathbf{M}$ is SDD and $m_{ii} \geq 0$, then $m_{ii} = |m_{ii}| > \sum_{j\neq i} |m_{ij}| = r_i^+(\mathbf{M}) + r_i^-(\mathbf{M}) \geq r_i^-(\mathbf{M}) \geq 0$, and $m_i^+ \geq m_i^- = m_{ii} - r_i^-(\mathbf{M}) > 0$. □

Proof of Theorem 2.2. Let $(\mathbf{x}, \lambda)$ be a solution of EiCP$(\mathbf{A}, \mathbf{B})$. Then $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{e}^\top \mathbf{x} = 1$ and there exists p with $x_p = \max_{i\in[n]} x_i > 0$. With $\mathbf{w} := (\mathbf{A} - \lambda\mathbf{B})\mathbf{x} \geq 0$ and $\mathbf{x}^\top \mathbf{w} = 0$, all terms $x_i w_i$, for $i \in [n]$, are nonnegative and sum to zero. Hence $x_p w_p = 0$, therefore $w_p = 0$ and $\lambda = \mathbf{A}_p^\top \mathbf{x}/\mathbf{B}_p^\top \mathbf{x}$.

Lemma 2.3 (i) provides bounds to the operands: $a_p^- x_p \leq \mathbf{A}_p^\top \mathbf{x} \leq a_p^+ x_p$ and $b_p^- x_p \leq \mathbf{B}_p^\top \mathbf{x} \leq b_p^+ x_p$, with the bounds on the denominator being both positive since $\mathbf{B}$ is SDD and $\mathbf{B}$ is PD then $b_{pp} = \mathbf{e}_p^\top \mathbf{B}\mathbf{e}_p > 0$. Thus, $a_p^- x_p/\mathbf{B}_p^\top \mathbf{x} \leq \lambda \leq a_p^+ x_p/\mathbf{B}_p^\top \mathbf{x}$.

The map $u \to v/u$ is monotonic on $\mathbb{R}_+$, then $a_p^- x_p/\mathbf{B}_p^\top \mathbf{x} \geq \min(a_p^- x_p/b_p^- x_p, a_p^- x_p/b_p^+ x_p) = \min(a_p^-/b_p^-, a_p^-/b_p^+)$, and, similarly, $a_p^+ x_p/\mathbf{B}_p^\top \mathbf{x} \leq \max(a_p^+/b_p^-, a_p^+/b_p^+)$ as $x_p \leq 1$.

The localization set K_1 is expressed as the union of such intervals for every possible value of $p \in [n]$, all intervals being well defined since $b_i^+ \geq b_i^- > 0$ for all $i \in [n]$.

□

Corollary 2.4. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}_n$. Assume that $\mathbf{B}$ is positive definite and strictly diagonally dominant, and $\mathbf{A}$ is copositive. Let $\lambda \in \mathbb{R}$ be a complementarity eigenvalue of $(\mathbf{A}, \mathbf{B})$, then*

$$\lambda \in \bigcup_{i \in [n]} \left[\max\left(0, \frac{a_i^-}{b_i^+}\right), \frac{a_i^+}{b_i^-} \right] =: K_1'.$$

Proof. The copositivity of the matrix $\mathbf{A}$ allows refining the localization set K_1 by considering the following properties.

(i) Property 1.5 establishes a trivial lower bound of zero for λ. The lower bounds of each interval in K_1 depend on the sign of a_i^-. With a negative a_i^-, the corresponding lower bound is also negative and thus dominated by zero. On the contrary, a tighter lower bound can be obtained when $a_i^- > 0$. Since the copositivity of the matrix alone does not induce a sign on a_i^-, we redefine the lower bound of each interval in K_1'.

(ii) Since $a_i^+ \geq 0$ and $b_i^+ \geq b_i^- > 0$ the upper bound of each interval in K_1 is simply a_i^+/b_i^-.

□

We now introduce an alternative localization set, which considers pairs of rows $i, j \in [n]$ of the matrices $\mathbf{A}$ and $\mathbf{B}$. As before, we compute quantities based on the off-diagonal entries. Moreover, to derive this new localization set, we make the additional assumption that $\mathbf{A}$ is copositive, then by Property 1.5 one has $\lambda \geq 0$. Note that the copositivity hypothesis of $\mathbf{A}$ is not restrictive since it can be enforced by a shift, as shown in Property 1.4.

We begin with the following definitions.

Definition 2.5. For any $\mathbf{M} \in \mathbb{R}^{n \times n}$, and two rows $i, j \in [n]$, let us denote:

$$m_{ij}^+ := m_{ii} m_{jj} - r_i^+(\mathbf{M}) r_j^+(\mathbf{M})$$
$$m_{ij}^- := m_{ii} m_{jj} - r_i^-(\mathbf{M}) r_j^-(\mathbf{M}).$$

Definition 2.6. For any pair $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ and two rows $i, j \in [n]$, let us denote:

$$s_{ij}^+(\mathbf{A}, \mathbf{B}) := a_{ii} b_{jj} + a_{jj} b_{ii} + r_i^+(\mathbf{A}) r_j^-(\mathbf{B}) + r_j^+(\mathbf{A}) r_i^-(\mathbf{B})$$
$$s_{ij}^-(\mathbf{A}, \mathbf{B}) := a_{ii} b_{jj} + a_{jj} b_{ii} + r_i^-(\mathbf{A}) r_j^+(\mathbf{B}) + r_j^-(\mathbf{A}) r_i^+(\mathbf{B})$$
$$P_{ij}^{\text{up}}(y) := b_{ij}^- y^2 - s_{ij}^+(\mathbf{A}, \mathbf{B}) y + a_{ij}^+, \quad P_{ij}^{\text{low}}(y) := b_{ij}^+ y^2 - s_{ij}^-(\mathbf{A}, \mathbf{B}) y + a_{ij}^- \quad \forall y \in \mathbb{R}.$$

Theorem 2.7 (Two-row localization)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}_n$. Assume that $\mathbf{B}$ is positive definite and strictly diagonally dominant, and $\mathbf{A}$ is copositive. Let $\lambda \in \mathbb{R}$ be a complementarity eigenvalue of $(\mathbf{A}, \mathbf{B})$, then*

$$\lambda \in \bigcup_{\substack{i,j \in [n] \\ i \neq j}} \left[\max\{0, \min\{y \in \mathbb{R} \mid P_{ij}^{\text{low}}(y) = 0\}\}, \ \max\{y \in \mathbb{R} \mid P_{ij}^{\text{up}}(y) = 0\} \right] =: K_2.$$

The proof of Theorem 2.7 makes use of the following results.

Lemma 2.8. *Let* $\mathbf{M} \in \mathbb{S}_n$ *and* $\mathbf{x} \in \mathbb{R}^n$ *with* $\mathbf{x} \geq \mathbf{0}$ *and* $\mathbf{e}^\top \mathbf{x} = 1$*, and the two largest elements*

$$p \in \operatorname{argmax}_{i\in[n]} x_i \text{ and } q \in \operatorname{argmax}_{i\in[n]\setminus\{p\}} x_i$$

then the following bounds hold for $i = p$ *and for* $i = q$*:*

$$-r_i^-(\mathbf{M})x_p x_q \;\leq\; \sum_{j\neq i} m_{ij} x_i x_j \;\leq\; r_i^+(\mathbf{M})x_p x_q,$$

Proof. Because $\mathbf{x} \geq \mathbf{0}$ and $x_p \geq x_q \geq x_j$ for $j \notin \{p,q\}$, we have for the p-row

$$\sum_{j\neq p} m_{pj}x_p x_j = \sum_{\substack{j\neq p\\ m_{pj}>0}} m_{pj}x_p x_j + \sum_{\substack{j\neq p\\ m_{pj}<0}} m_{pj}x_p x_j \;\leq\; \sum_{\substack{j\neq p\\ m_{pj}>0}} m_{pj}x_p x_q = r_p^+(\mathbf{M})x_p x_q,$$

and, symmetrically,

$$\sum_{j\neq p} m_{pj}x_p x_j \;\geq\; \sum_{\substack{j\neq p\\ m_{pj}<0}} m_{pj}x_p x_j \;\geq\; -\sum_{\substack{j\neq p\\ m_{pj}<0}} |m_{pj}|x_p x_q = -r_p^-(\mathbf{M})x_p x_q.$$

For the case $i = q$, we use that $x_j \leq x_p$ for all $j \neq q$, then

$$\sum_{j\neq q} m_{qj}x_q x_j \leq \sum_{\substack{j\neq q\\ m_{qj}>0}} m_{qj}x_q x_p = r_q^+(\mathbf{M})x_p x_q, \quad \sum_{j\neq q} m_{qj}x_q x_j \geq \sum_{\substack{j\neq q\\ m_{qj}<0}} m_{qj}x_q x_p = -r_q^-(\mathbf{M})x_p x_q.$$

□

Lemma 2.9. *The quadratic functions* P_{ij}^{low} *and* P_{ij}^{up} *presented in Definition 2.6 can also be written as follows, for all* $y \in \mathbb{R}$*,*

$$P_{ij}^{\text{up}}(y) = (yb_{ii} - a_{ii})(yb_{jj} - a_{jj}) - (r_i^+(\mathbf{A}) + yr_i^-(\mathbf{B}))(r_j^+(\mathbf{A}) + yr_j^-(\mathbf{B})), \tag{2.1}$$

$$P_{ij}^{\text{low}}(y) = (yb_{ii} - a_{ii})(yb_{jj} - a_{jj}) - (r_i^-(\mathbf{A}) + yr_i^+(\mathbf{B}))(r_j^-(\mathbf{A}) + yr_j^+(\mathbf{B})). \tag{2.2}$$

Proof. The definition of $P_{ij}^{\text{up}}(y)$ can be retrieved by expanding the expression (2.1):

$$\begin{aligned}
&\big(b_{ii}b_{jj} - r_i^-(\mathbf{B})r_j^-(\mathbf{B})\big)y^2 + \big(a_{ii}a_{jj} - r_i^+(\mathbf{A})r_j^+(\mathbf{A})\big)\\
&\quad - \Big(a_{ii}b_{jj} + a_{jj}b_{ii} + r_i^+(\mathbf{A})r_j^-(\mathbf{B}) + r_j^+(\mathbf{A})r_i^-(\mathbf{B})\Big)y\\
&= b_{ij}^- y^2 - s_{ij}^+(\mathbf{A},\mathbf{B})y + a_{ij}^+ = P_{ij}^{\text{up}}(y)
\end{aligned}$$

Note that, due to symmetry in Definition 2.6, $P_{ij}^{\text{low}}(y)$ in (2.2) is obtained from $P_{ij}^{\text{up}}(y)$ in (2.1) only by replacing the terms r^+ with r^- and vice versa.

□

Lemma 2.10. *Under the hypotheses of Theorem 2.7, the quadratic functions* P_{ij}^{low} *and* P_{ij}^{up} *are strictly convex and have real roots. As a consequence, for any* $y \in \mathbb{R}$*, if* $P_{ij}^{\text{up}}(y) \leq 0$*, then* $y \leq \max\{y \in \mathbb{R} \mid P_{ij}^{\text{up}}(y) = 0\}$*, or if* $P_{ij}^{\text{low}}(y) \leq 0$*, then* $y \geq \min\{y \in \mathbb{R} \mid P_{ij}^{\text{low}}(y) = 0\}$*.*

Proof. According to Lemma 2.3 (ii), the leading coefficients b_{ij}^- and b_{ij}^+ of the quadratic polynomials are positive, thus P_{ij}^{low} and P_{ij}^{up} are strictly convex. For a strictly convex quadratic polynomial P with two real roots (not necessarily distinct), the interval in-between is precisely $\{y \in \mathbb{R} : P(y) \leq 0\}$. Thus, it remains to show that P_{ij}^{low} and P_{ij}^{up} have nonnegative discriminants Δ_{ij}^{low} and Δ_{ij}^{up}.

Since $\mathbf{A}$ is copositive and $\mathbf{B}$ is PD, their diagonal terms are nonnegative, so are the one-row sums r^+ and r^- from Definition 2.1, hence:

$$a_{ii}b_{jj}+a_{jj}b_{ii} \geq 2\sqrt{a_{ii}a_{jj}b_{ii}b_{jj}}, \quad r_i^-(\mathbf{A})r_j^+(\mathbf{B})+r_j^-(\mathbf{A})r_i^+(\mathbf{B}) \geq 2\sqrt{r_i^-(\mathbf{A})r_j^-(\mathbf{A})\,r_i^+(\mathbf{B})r_j^+(\mathbf{B})}.$$

By summing and squaring each side, we get

$$s_{ij}^-(\mathbf{A},\mathbf{B})^2 \;\geq\; \left(2\sqrt{a_{ii}a_{jj}b_{ii}b_{jj}}+2\sqrt{r_i^-(\mathbf{A})r_j^-(\mathbf{A})\,r_i^+(\mathbf{B})r_j^+(\mathbf{B})}\right)^2.$$

Subtracting $4b_{ij}^+a_{ij}^- = 4\big(b_{ii}b_{jj}-r_i^+(\mathbf{B})r_j^+(\mathbf{B})\big)\big(a_{ii}a_{jj}-r_i^-(\mathbf{A})r_j^-(\mathbf{A})\big)$ yields

$$\Delta_{ij}^{\text{low}} := s_{ij}^-(\mathbf{A},\mathbf{B})^2-4b_{ij}^+a_{ij}^- \;\geq\; 4\left(\sqrt{a_{ii}a_{jj}\,r_i^+(\mathbf{B})r_j^+(\mathbf{B})}+\sqrt{b_{ii}b_{jj}\,r_i^-(\mathbf{A})r_j^-(\mathbf{A})}\right)^2 \;\geq\; 0.$$

The proof is symmetric for Δ_{ij}^{up} by inverting r^+ and r^- as observed in Lemma 2.9, i.e.,

$$\Delta_{ij}^{\text{up}} := s_{ij}^+(\mathbf{A},\mathbf{B})^2-4b_{ij}^-a_{ij}^+ \;\geq\; 4\left(\sqrt{a_{ii}a_{jj}\,r_i^-(\mathbf{B})r_j^-(\mathbf{B})}+\sqrt{b_{ii}b_{jj}\,r_i^+(\mathbf{A})r_j^+(\mathbf{A})}\right)^2 \;\geq\; 0.$$

□

Proof of Theorem 2.7. Let $(\mathbf{x},\lambda)$ be a complementarity eigenpair of $(\mathbf{A},\mathbf{B})$. The proof is split into two cases, depending on whether the vector $\mathbf{x}$ has one nonzero value (i.e., $\mathbf{x}=\mathbf{e}_p$ for some index $p\in[n]$) or at least two.

Case 1: $\mathbf{x}=\mathbf{e}_p$, $p\in[n]$. Due to the complementarity constraints, the p-row of (1.1a) reads as $\lambda b_{pp}-a_{pp}=0$, that is, $\lambda = a_{pp}/b_{pp}$ which is nonnegative due to the positivity assumptions on $\mathbf{A}$ and $\mathbf{B}$. For any index $j\in[n]$, with $j\neq p$, according to Lemma 2.9, P_{pj}^{up} and P_{pj}^{low} evaluate at $\lambda = a_{pp}/b_{pp}$ as

$$P_{pj}^{\text{up}}(\lambda) = -\Big(r_p^+(\mathbf{A})+\lambda\, r_p^-(\mathbf{B})\Big)\Big(r_j^+(\mathbf{A})+\lambda\, r_j^-(\mathbf{B})\Big)$$
$$P_{pj}^{\text{low}}(\lambda) = -\Big(r_p^-(\mathbf{A})+\lambda\, r_p^+(\mathbf{B})\Big)\Big(r_j^-(\mathbf{A})+\lambda\, r_j^+(\mathbf{B})\Big),$$

which are both nonpositive, since λ, r^+, and r^- are all nonnegative. By Lemma 2.10, λ belongs to at least $n-1$ intervals (with $i=p$) defining K_2.

Case 2: $\mathbf{x}_p\geq\mathbf{x}_q>0$, $p\neq q$. From now on, assume that $\mathbf{x}$ has at least two positive elements and p and q are distinct indices in $[n]$ of the largest and second largest elements of $\mathbf{x}$ as in Lemma 2.8. The complementarity condition (1.1c) reads as

$$(\lambda b_{ii}-a_{ii})x_i^2 = \sum_{j\neq i}(a_{ij}-\lambda b_{ij})x_ix_j, \text{ for } i\in[n]. \tag{2.3}$$

Applying bounds from Lemma 2.8 to this identity for $i=p$ or $i=q$, and using $\lambda\geq 0$ by Property 1.5 yield

$$(\lambda b_{ii}-a_{ii})x_i^2 \;\leq\; \big(r_i^+(\mathbf{A})+\lambda r_i^-(\mathbf{B})\big)\,x_px_q, \text{ for } i\in\{p,q\}. \tag{2.4}$$

If $\lambda > a_{ii}/b_{ii}$ for both $i\in\{p,q\}$, then left-hand sides of inequalities (2.4) are positive. Multiplying the two inequalities and dividing by $x_p^2x_q^2>0$ give

$$(\lambda b_{pp}-a_{pp})(\lambda b_{qq}-a_{qq}) \;\leq\; (r_p^+(\mathbf{A})+\lambda\, r_p^-(\mathbf{B}))\,(r_q^+(\mathbf{A})+\lambda\, r_q^-(\mathbf{B})).$$

By rearranging all the terms to the left side and by using (2.1), we have $P_{pq}^{\text{up}}(\lambda)\leq 0$.

We consider now that $\lambda \leq a_{ii}/b_{ii}$ for some $i \in \{p,q\}$. From Case 1, we have that $P^{\text{up}}_{pq}(a_{ii}/b_{ii}) \leq 0$, which allows concluding that $\lambda \leq a_{ii}/b_{ii} \leq \max\{y \in \mathbb{R} \mid P^{\text{up}}_{pq}(y) = 0\}$.

Symmetric arguments apply to prove that $\lambda \geq \min\{y \in \mathbb{R} \mid P^{\text{low}}_{pq}(y) = 0\}$. We first multiply both sides of identity (2.3) by -1, then we apply Lemma 2.8 with $\lambda \geq 0$, yielding

$$(a_{ii} - \lambda b_{ii})x_i^2 \leq \left(\lambda r_i^+(\mathbf{B}) + r_i^-(\mathbf{A})\right) x_p x_q, \text{ for } i \in \{p,q\}. \tag{2.5}$$

If $\lambda < a_{ii}/b_{ii}$ in (2.5) for both $i \in \{p,q\}$ then these inequalities have positive left-hand sides; their product reads $P^{\text{low}}_{pq}(\lambda) \leq 0$, and then $\lambda \geq \min\{y \in \mathbb{R} \mid P^{\text{low}}_{pq}(y) = 0\}$. If $\lambda \geq a_{ii}/b_{ii}$ for some $i \in \{p,q\}$, then $P^{\text{low}}_{pq}(a_{ii}/b_{ii}) \leq 0$, and, by transitivity, $\lambda \geq a_{ii}/b_{ii} \geq \min\{y \in \mathbb{R} \mid P^{\text{low}}_{pq}(y) = 0\}$.

Thus we have shown that λ belongs to at least one interval, with $(i,j) = (p,q)$, of K_2.

□

Remark 2.11. From Case 1 in the latter proof, we see that each interval in the union defining K_2 is well-defined for every pair of indices $i, j \in [n]$, as it contains the values a_{ii}/b_{ii} and a_{jj}/b_{jj}. Indeed, applying Lemma 2.9 at $y = a_{ii}/b_{ii}$, which is nonnegative by hypothesis, yields $P^{\text{low}}_{ij}(a_{ii}/b_{ii}) \leq 0$ and $P^{\text{up}}_{ij}(a_{ii}/b_{ii}) \leq 0$, then, from Lemma 2.10,

$$\min\{y \in \mathbb{R} \mid P^{\text{low}}_{ij}(y) = 0\} \leq a_{ii}/b_{ii} \leq \max\{y \in \mathbb{R} \mid P^{\text{up}}_{ij}(y) = 0\}.$$

3. Comparing the localization sets

In this section, we show that the two-row set K_2 presented in Theorem 2.7 is smaller than the one-row set K_1 given in Theorem 2.2. At this aim, it is necessary to consider the same assumptions for both statements (in particular, the copositivity of $\mathbf{A}$ in K_1). Note that $K_1' \subset K_1$ (by construction) and then prove that $K_2 \subset K_1'$ directly implies $K_2 \subset K_1$. Therefore, we introduce below the notations C^{up}_{ij} and C^{low}_{ij} representing the two-row version of the endpoints from the intervals, which define K_1' in Corollary 2.4.

Lemma 3.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}_n$. Assume that $\mathbf{B}$ is positive definite and strictly diagonally dominant, and $\mathbf{A}$ is copositive. For any $i, j \in [n]$, such that $i \neq j$, let us define*

$$C^{\text{up}}_{ij} := \max\left\{\frac{a_i^+}{b_i^-}, \frac{a_j^+}{b_j^-}\right\}, \quad C^{\text{low}}_{ij} := \min\left\{\frac{a_i^-}{b_i^+}, \frac{a_j^-}{b_j^+}\right\}, \tag{3.1}$$

$$y^{\star,\text{up}}_{ij} := \frac{s^+_{ij}(\mathbf{A},\mathbf{B})}{2b^-_{ij}}, \quad y^{\star,\text{low}}_{ij} := \frac{s^-_{ij}(\mathbf{A},\mathbf{B})}{2b^+_{ij}}. \tag{3.2}$$

Then, the following properties hold

(a) $y^{\star,\text{up}}_{ij}$ and $y^{\star,\text{low}}_{ij}$ are the vertices of P^{up}_{ij} and P^{low}_{ij}, respectively.

(b) $y^{\star,\text{up}}_{ij} \leq C^{\text{up}}_{ij}$ and $y^{\star,\text{low}}_{ij} \geq C^{\text{low}}_{ij}$ for every $i, j \in [n]$, such that $i \neq j$.

Proof. To prove the statement in (a), recall from Lemma 2.10 that P^{up}_{ij} is a strictly convex quadratic function. Its minimum occurs at the vertex of this parabola, which can be found by setting the first derivative equal to zero. This gives $y^{\star,\text{up}}_{ij}$ in (3.2). Similarly, $y^{\star,\text{low}}$ in (3.2) is the vertex of P^{low}_{ij}.

The two inequalities in (b) can be obtained by the following reasoning.

(i) From the definition of C_{ij}^{up} and $b_i^- > 0$ and $b_j^- > 0$ as $\mathbf{B}$ is PD and SDD:

$$a_i^+ \leq C_{ij}^{\text{up}} b_i^-, \quad a_j^+ \leq C_{ij}^{\text{up}} b_j^-.$$

Multiplying the first inequality by $b_{jj} + r_j^-(\mathbf{B})$ and the second one by $b_{ii} + r_i^-(\mathbf{B})$ and adding the two resulting inequalities yield

$$a_i^+(b_{jj} + r_j^-(\mathbf{B})) + a_j^+(b_{ii} + r_i^-(\mathbf{B})) \;\leq\; C_{ij}^{\text{up}}\left(b_i^-(b_{jj} + r_j^-(\mathbf{B})) + b_j^-(b_{ii} + r_i^-(\mathbf{B}))\right).$$

The right-hand side equals $2C_{ij}^{\text{up}} b_{ij}^-$ and expanding the left-hand side gives

$$s_{ij}^+(\mathbf{A},\mathbf{B}) \;+\; a_{ii} r_j^-(\mathbf{B}) + a_{jj} r_i^-(\mathbf{B}) + r_i^+(\mathbf{A}) b_{jj} + r_j^+(\mathbf{A}) b_{ii},$$

where all terms are nonnegative. Thus $s_{ij}^+(\mathbf{A},\mathbf{B}) \leq 2C_{ij}^{\text{up}} b_{ij}^-$, which yields

$$y_{ij}^{\star,\text{up}} := \frac{s_{ij}^+(\mathbf{A},\mathbf{B})}{2b_{ij}^-} \leq C_{ij}^{\text{up}}.$$

(ii) Similarly, as $b^- > 0$ we have

$$a_i^-(b_{jj} - r_j^+(\mathbf{B})) + a_j^-(b_{ii} - r_i^+(\mathbf{B})) \;\geq\; C_{ij}^{\text{low}}\left(b_i^+(b_{jj} - r_j^+(\mathbf{B})) + b_j^+(b_{ii} - r_i^+(\mathbf{B}))\right)$$

which yields

$$s_{ij}^-(\mathbf{A},\mathbf{B}) \geq s_{ij}^-(\mathbf{A},\mathbf{B}) \;-\; a_{ii} r_j^+(\mathbf{B}) - a_{jj} r_i^+(\mathbf{B}) - r_i^-(\mathbf{A}) b_{jj} - r_j^-(\mathbf{A}) b_{ii} \geq 2C_{ij}^{\text{low}} b_{ij}^+,$$

hence $y_{ij}^{\star,\text{low}} \geq C_{ij}^{\text{low}}$.

□

In the proof of the following theorem, we can assume without loss of generality that $C_{ij}^{\text{low}} > 0$ because, otherwise, from the definition of K_1' in Corollary 2.4, the lower bound of the interval associated with (i,j) is zero (which is always equal to or less than any endpoint of the intervals that define K_2).

Theorem 3.2 (K_2 is tighter than K_1)**.** *Let $\mathbf{A},\mathbf{B} \in \mathbb{S}_n$. Assume that $\mathbf{B}$ is positive definite and strictly diagonally dominant, and that $\mathbf{A}$ is copositive, then $K_2 \subset K_1$ (precisely $K_2 \subset K_1' \subset K_1$).*

Proof. For any $i \in [n]$ and $y \in \mathbb{R}$, let us denote

$$X_i(y) = b_{ii} y - a_{ii}, \quad U_i(y) = r_i^+(\mathbf{A}) + y r_i^-(\mathbf{B}), \quad L_i(y) = r_i^-(\mathbf{A}) + y r_i^+(\mathbf{B}).$$

By Lemma 2.9, for $i,j \in [n], i \neq j$, we have

$$P_{ij}^{\text{up}}(y) = X_i(y)X_j(y) - U_i(y)U_j(y), \quad P_{ij}^{\text{low}}(y) = X_i(y)X_j(y) - L_i(y)L_j(y). \tag{3.3}$$

Furthermore, $U_i(y) \geq 0$ and $L_i(y) \geq 0$ for $y \geq 0$ and we have the equivalences:

$$X_i(y) > U_i(y) \iff (b_{ii} - r_i^-(\mathbf{B}))\,y > a_{ii} + r_i^+(\mathbf{A}) \iff y > a_i^+/b_i^-, \tag{3.4}$$

$$-X_i(y) > L_i(y) \iff (b_{ii} + r_i^+(\mathbf{B}))\,y < a_{ii} - r_i^-(\mathbf{A}) \iff y < a_i^-/b_i^+. \tag{3.5}$$

- If $y > \max\left\{\frac{a_i^+}{b_i^-}, \frac{a_j^+}{b_j^-}\right\} = C_{ij}^{\text{up}}$, then $X_i(y) > U_i(y) \geq 0$ and $X_j(y) > U_j(y) \geq 0$. By using the definition of P_{ij}^{up} in (3.3), these conditions yield to $P_{ij}^{\text{up}}(y) > 0$. Since P_{ij}^{up} is a strictly convex quadratic function, the values of y such that P_{ij}^{up} is positive lie beyond its smallest and largest root. However, from Lemma 3.1, we get $y > C_{ij}^{\text{up}} \geq y_{ij}^{\star,\text{up}}$, that is y lies to the *right* of the largest root of P_{ij}^{up}.

- If $0 \leq y < \min\left\{\dfrac{a_i^-}{b_i^+}, \dfrac{a_j^-}{b_j^+}\right\} = C_{ij}^{\text{low}}$, then $-X_i(y) > L_i(y) \geq 0$ and $-X_j(y) > L_j(y) \geq 0$, hence $P_{ij}^{\text{low}}(y) > 0$. Again, from Lemma 3.1, y lies to the *left* of the vertex $y_{ij}^{\star,\text{low}}$ of P_{ij}^{low}, and to its smallest root.

Since the endpoints that define K_1' are $\dfrac{a_i^-}{b_i^+}$ and $\dfrac{a_i^+}{b_i^-}$, for $i \in [n]$, and the endpoints that define K_2 are the smallest roots of P_{ij}^{low} and largest roots of P_{ij}^{up}, for $i \neq j$, the two items above imply that every interval contributing to K_2 is contained in some interval contributing to K_1'. Taking the union over all ordered pairs (i, j) with $i \neq j$ yields $K_2 \subset K_1'$ implying $K_2 \subset K_1$.

□

4. Lower bounds and upper bounds

The localization sets K_1 and K_2 constrain each complementarity eigenvalue, which is stronger than giving only global lower or upper bounds. For practical screening and benchmarking, it is often enough to control the extremes of the complementarity spectrum. We therefore extract computable bounds for the largest and the smallest complementarity eigenvalues, first in the one-row case and then in the two-row case, as stated in Corollaries 4.1 and 4.2.

Corollary 4.1 (One-row bounds)**.** *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}_n$*, and assume that* $\mathbf{B}$ *is positive definite and strictly diagonally dominant. Let* λ *be a complementarity eigenvalue of* $(\mathbf{A}, \mathbf{B})$*. Then,*

$$\min\left\{\min_{i\in[n]} \frac{a_i^-}{b_i^-}, \min_{i\in[n]} \frac{a_i^-}{b_i^+}\right\} \leq \lambda \leq \max\left\{\max_{i\in[n]} \frac{a_i^+}{b_i^-}, \max_{i\in[n]} \frac{a_i^+}{b_i^+}\right\}.$$

Corollary 4.2 (Two-row bounds)**.** *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}_n$*. Assume that* $\mathbf{B}$ *is positive definite and strictly diagonally dominant, and* $\mathbf{A}$ *is copositive. Let* λ *be a complementarity eigenvalue of* $(\mathbf{A}, \mathbf{B})$*. Then,*

$$\min_{\substack{i,j\in[n]\\ i\neq j}} \frac{s_{ij}^-(\mathbf{A},\mathbf{B}) - \sqrt{s_{ij}^-(\mathbf{A},\mathbf{B})^2 - 4\,b_{ij}^+ a_{ij}^-}}{2\,b_{ij}^+} \leq \lambda \leq \max_{\substack{i,j\in[n]\\ i\neq j}} \frac{s_{ij}^+(\mathbf{A},\mathbf{B}) + \sqrt{s_{ij}^+(\mathbf{A},\mathbf{B})^2 - 4\,b_{ij}^- a_{ij}^+}}{2\,b_{ij}^-}.$$

It is possible to interpret both bounds above as the *convex hull* of K_1 and K_2. In other words, Corollaries 4.1 and 4.2 can be rewritten as $\lambda \in \operatorname{conv}(K_1)$ and $\lambda \in \operatorname{conv}(K_2)$, respectively.

Example 4.3. Consider the matrices in $\mathbb{R}^3$ given by

$$\mathbf{A} = \begin{pmatrix} 14 & 1 & 1 \\ 1 & 11 & -2 \\ 1 & -2 & 13 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & 2 & 10 \end{pmatrix}.$$

Note that both matrices are SDDs with positive diagonals, so they are both PD. In particular, $\mathbf{A}$ is copositive. Then, the assumptions of Theorem 2.2 and Theorem 2.7 hold.

As described in the introduction, we can enumerate complementarity eigenpairs by supports $S \subset \{1,2,3\}$: for each nonempty S, candidates λ are the generalized eigenvalues of $(\mathbf{A}_{SS}, \mathbf{B}_{SS})$

with $\mathbf{x}_S \geq \mathbf{0}$, then verified by the off-support inequalities $(\mathbf{A}_{iS} - \lambda \mathbf{B}_{iS})\mathbf{x} \geq \mathbf{0}$ when $i \notin S$. Feasible supports are $\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$, and the complementarity spectrum is

$$\Pi \approx \{0.822,\ 2.333,\ 2.347,\ 2.349,\ 2.352\}.$$

(i) The one-row enclosure from Corollary 4.1 gives

$$\Pi \subset \operatorname{conv}(K_1) = \left[\frac{3}{4}, \frac{8}{3}\right] \approx [0.750,\ 2.667].$$

(ii) The two-row enclosure from Corollary 4.2 reduces to

$$\Pi \subset \operatorname{conv}(K_2) = \left[\frac{31 - \sqrt{127}}{24}, \frac{109 + \sqrt{1081}}{60}\right] \approx [0.822,\ 2.365].$$

As shown in Theorem 3.2, we have that $\operatorname{conv}(K_2) \subset \operatorname{conv}(K_1)$.

Let us compare this with the spectrum $[\mu_{\min}(\mathbf{A},\mathbf{B}), \mu_{\max}(\mathbf{A},\mathbf{B})]$ of the classical generalized eigenvalues. The smallest and largest generalized eigenvalues of two symmetric matrices $(\mathbf{A},\mathbf{B})$ and with $\mathbf{B}$ being PD can be defined as the solution of minimization and maximization problems of the generalized Rayleigh quotient (1.2) for $\mathbf{x} \neq \mathbf{0}$ (see Sect. A.5.3 in [3]), that is

$$\mu_{\min}(\mathbf{A},\mathbf{B}) := \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A}\,\mathbf{x}}{\mathbf{x}^\top \mathbf{B}\,\mathbf{x}}, \quad \mu_{\max}(\mathbf{A},\mathbf{B}) := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A}\,\mathbf{x}}{\mathbf{x}^\top \mathbf{B}\,\mathbf{x}}. \tag{4.1}$$

Then, we have $\Pi \subset [\mu_{\min}(\mathbf{A},\mathbf{B}), \mu_{\max}(\mathbf{A},\mathbf{B})] \approx [0.804, 2.352]$. Hence, in this instance, the generalized spectrum is also a localization set for the complementarity eigenvalues, which is contained in $\operatorname{conv}(K_1)$ but only overlaps $\operatorname{conv}(K_2)$.

5. Comparison with the generalized spectrum

In this section, we show that this last remark is not incidental: the smallest and largest generalized eigenvalues always provide lower and upper bounds for the complementary eigenvalues of (1.1), and they are not comparable with the bounds of Section 4.

Proposition 5.1 (Generalized spectral localization set)**.** *Let $\mathbf{A},\mathbf{B} \in \mathbb{S}_n$ and assume that $\mathbf{B}$ is positive definite. If λ is a complementarity eigenvalue of $(\mathbf{A},\mathbf{B})$ and $\mu_{\min}(\mathbf{A},\mathbf{B})$ and $\mu_{\max}(\mathbf{A},\mathbf{B})$ are the minimum and maximum generalized eigenvalues of $(\mathbf{A},\mathbf{B})$, then*

$$\lambda \in [\mu_{\min}(\mathbf{A},\mathbf{B}), \mu_{\max}(\mathbf{A},\mathbf{B})] := \Gamma$$

Proof. Let $(\mathbf{x},\lambda)$ be a solution of EiCP (1.1). The orthogonality constraint (1.1c) yields $\lambda = \dfrac{\mathbf{x}^\top \mathbf{A}\,\mathbf{x}}{\mathbf{x}^\top \mathbf{B}\,\mathbf{x}}$, which is well defined as $\mathbf{B}$ is PD and $\mathbf{x} \neq \mathbf{0}$. By using the definitions in (4.1), we get the result. □

To show that dominance does not hold between Γ and the sets K_1 and K_2, we rely on the following lemma, which characterizes the generalized eigenvalues when $\mathbf{A}$ and $\mathbf{B}$ commute. We then derive two families of problems where the generalized spectrum is either strictly looser (Proposition 5.3) or strictly tighter (Proposition 5.4) than our localization sets.

Lemma 5.2. *Let $\mathbf{A},\mathbf{B}\in\mathbb{S}_n$, assume that $\mathbf{B}$ is positive definite and $\mathbf{AB}=\mathbf{BA}$. Let $\mu_1^{\mathbf{A}},\dots,\mu_n^{\mathbf{A}}$ be the standard eigenvalues of $\mathbf{A}$ and let $\mu_1^{\mathbf{B}},\dots,\mu_n^{\mathbf{B}}>0$ be those of $\mathbf{B}$, all counted with multiplicity. Then, the generalized eigenvalues $\mu_1,\dots,\mu_n$ of the pair $(\mathbf{A},\mathbf{B})$ are exactly the n ratios*

$$\mu_1=\mu^{\mathbf{A}}_{\sigma(1)}/\mu^{\mathbf{B}}_{\tau(1)},\ \dots,\ \mu_n=\mu^{\mathbf{A}}_{\sigma(n)}/\mu^{\mathbf{B}}_{\tau(n)}$$

for some permutations σ,τ of $[n]$. By indexing a common orthonormal eigenbasis, one can take σ and τ as the identity, to obtain $\mu_i=\mu_i^{\mathbf{A}}/\mu_i^{\mathbf{B}}$ for $i\in[n]$.

Proof. By the fact that commuting real symmetric matrices are simultaneously diagonalizable by an orthogonal matrix (see Thm. 2.5.5 in [14]), there exists an orthogonal $\mathbf{V}\in\mathbb{R}^{n\times n}$ such that

$$\mathbf{V}^\top\mathbf{A}\,\mathbf{V}=\operatorname{diag}(\mu_1^{\mathbf{A}},\dots,\mu_n^{\mathbf{A}}),\quad \mathbf{V}^\top\mathbf{B}\,\mathbf{V}=\operatorname{diag}(\mu_1^{\mathbf{B}},\dots,\mu_n^{\mathbf{B}}).$$

Consider the equation $\mathbf{Ax}=\mu\,\mathbf{Bx}$ for any generalized eigenvalue μ. Setting $\mathbf{y}=\mathbf{V}^\top\mathbf{x}$ yields

$$(\mathbf{V}^\top\mathbf{A}\,\mathbf{V})\mathbf{y}=\mu\,(\mathbf{V}^\top\mathbf{B}\,\mathbf{V})\mathbf{y}\iff(\mu_i^{\mathbf{A}}-\mu\,\mu_i^{\mathbf{B}})\,y_i=0\quad\text{for all } i\in[n].$$

If $\mathbf{y}\neq\mathbf{0}$, then some $y_i\neq 0$ enforces $\mu=\mu_i^{\mathbf{A}}/\mu_i^{\mathbf{B}}$. Conversely, for each $i\in[n]$, taking $\mathbf{y}=\mathbf{e}_i$ (hence, $\mathbf{x}=\mathbf{Ve}_i$) produces a nonzero solution $\mu_i=\mu_i^{\mathbf{A}}/\mu_i^{\mathbf{B}}$. Thus the generalized eigenvalues are precisely the n ratios $\mu_i^{\mathbf{A}}/\mu_i^{\mathbf{B}}$ (counted with multiplicity).

□

Proposition 5.3 (A family where $K_1=K_2\subsetneq\Gamma$)**.** *Fix $n\geq 2$ and $\varepsilon>1$. Let $\mathbf{E}=\mathbf{e}\,\mathbf{e}^\top$, and define*

$$\mathbf{A}:=\mathbf{E}+\varepsilon\,\mathbf{I},\quad \mathbf{B}:=((n-1)+\varepsilon)\,\mathbf{I}-\mathbf{E}.$$

Then, $\mathbf{B}$ is SDD *and* PD*, and $\mathbf{A}$ is copositive (so Theorems 2.2 and 2.7 apply). Moreover,*

$$\operatorname{conv}(K_1)=\operatorname{conv}(K_2)=\left[\frac{1+\varepsilon}{n-2+\varepsilon}\ ,\ \frac{n+\varepsilon}{\varepsilon-1}\right],\quad [\mu_{\min}(\mathbf{A},\mathbf{B}),\mu_{\max}(\mathbf{A},\mathbf{B})]=\left[\frac{\varepsilon}{n-1+\varepsilon}\ ,\ \frac{n+\varepsilon}{\varepsilon-1}\right].$$

Hence, $K_1=K_2\subsetneq[\mu_{\min}(\mathbf{A},\mathbf{B}),\mu_{\max}(\mathbf{A},\mathbf{B})]$.

Proof. $\mathbf{B}$ is SDD because $b_{ii}=n-2+\varepsilon>\sum_{j\neq i}|b_{ij}|=n-1$ for $\varepsilon>1$, and $\mathbf{B}$ is symmetric with positive diagonal elements, thus $\mathbf{B}$ is PD. Moreover, $\mathbf{A}$ is nonnegative, thus it is copositive. The matrices $\mathbf{A}$ and $\mathbf{B}$ commute (as $\mathbf{AB}=\mathbf{BA}=\delta\mathbf{I}-\mathbf{E}$ for some constant δ) and from Lemma 5.2 we can compute the values of $\mu_{\min}$ and $\mu_{\max}$ by calculating the largest and smallest eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. To show that, we first recall the spectral properties of $\mathbf{E}$. Since $\mathbf{E}=\mathbf{ee}^\top$, it follows that $\mathbf{Ee}=n\mathbf{e}$ and $\mathbf{Ev}=0$ for all $\mathbf{v}$ such that $\mathbf{v}^\top\mathbf{e}=0$. Hence, $\mathbf{E}$ has eigenvalue n associated with the eigenvector $\mathbf{e}$, and eigenvalue 0 with multiplicity $n-1$ associated with any vector orthogonal to $\mathbf{e}$. Using this decomposition of $\mathbf{E}$, we can determine the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ below in (i) and (ii).

(i) For $\mathbf{A}=\mathbf{E}+\varepsilon\mathbf{I}$, we have

$$\mathbf{Ae}=(\mathbf{E}+\varepsilon\mathbf{I})\mathbf{e}=(n+\varepsilon)\mathbf{e},\quad \mathbf{Av}=(\mathbf{E}+\varepsilon\mathbf{I})\mathbf{v}=\varepsilon\mathbf{v}\quad\text{for every }\mathbf{v}\text{ such that }\mathbf{v}^\top\mathbf{e}=0.$$

Therefore, from the eigenvalue equations above, the classical spectrum of $\mathbf{A}$ is equal to $\{n+\varepsilon,\ \underbrace{\varepsilon,\dots,\varepsilon}_{n-1\text{ times}}\}$ and then $\mu^{\mathbf{A}}_{\min}=\varepsilon$, and $\mu^{\mathbf{A}}_{\max}=n+\varepsilon$.

(ii) For $\mathbf{B} = \big((n-1)+\varepsilon\big)\mathbf{I} - \mathbf{E}$,

$$\mathbf{B}\mathbf{e} = \big((n-1)+\varepsilon\big)\mathbf{e} - \mathbf{E}\mathbf{e} = \big((n-1)+\varepsilon-n\big)\mathbf{e} = (\varepsilon-1)\mathbf{e},$$

and for any $\mathbf{v}$ such that $\mathbf{v}^\mathsf{T}\mathbf{e} = 0$,

$$\mathbf{B}\mathbf{v} = \big((n-1)+\varepsilon\big)\mathbf{v} - \mathbf{E}\mathbf{v} = \big((n-1)+\varepsilon\big)\mathbf{v}.$$

Similarly, the spectrum of $\mathbf{B}$ is given by $\{\varepsilon-1, \underbrace{n-1+\varepsilon, \dots, n-1+\varepsilon}_{n-1 \text{ times}}\}$ and thus we have $\mu^{\mathbf{B}}_{\text{min}} = \varepsilon - 1$ and $\mu^{\mathbf{B}}_{\text{max}} = n-1+\varepsilon$.

Hence, $\mu_{\text{min}}(\mathbf{A},\mathbf{B}) = \dfrac{\varepsilon}{n-1+\varepsilon}$ and $\mu_{\text{max}}(\mathbf{A},\mathbf{B}) = \dfrac{n+\varepsilon}{\varepsilon-1}$.

For K_2, substitute $r_i^-(\mathbf{A}) = 0$, $r_i^+(\mathbf{A}) = n-1$, $r_i^-(\mathbf{B}) = n-1$, $r_i^+(\mathbf{B}) = 0$ into polynomial expressions of Lemma 2.9. This yields

$$P_{ij}^{\text{low}}(y) = \big((n-2+\varepsilon)y - (1+\varepsilon)\big)^2, \quad P_{ij}^{\text{up}}(y) = \big((n-2+\varepsilon)y - (1+\varepsilon)\big)^2 - (n-1)^2(1+y)^2,$$

so the intervals in K_2 are all equal to $\left[\dfrac{1+\varepsilon}{n-2+\varepsilon}, \dfrac{n+\varepsilon}{\varepsilon-1}\right]$ independently of (i,j). From Corollary 2.4, note that all intervals in K_1 also coincide with this. The upper bound coincides with μ_{max}, but the lower bound strictly improves upon μ_{min}, as:

$$\frac{1+\varepsilon}{n-2+\varepsilon} - \frac{\varepsilon}{n-1+\varepsilon} = \frac{(n-1)+2\varepsilon}{(n-2+\varepsilon)(n-1+\varepsilon)} > 0,$$

□

Proposition 5.4 (A family where $\Gamma \subsetneq K_1 = K_2$)**.** *Fix $n \geq 2$ and parameters $\beta > R > 0$ and $c > 0$. Let $\mathbf{E} = \mathbf{e}\mathbf{e}^\top$, set $\rho := R/(n-1)$, and define*

$$\mathbf{B} := \beta\,\mathbf{I} + \rho\,(\mathbf{E}-\mathbf{I}), \quad \mathbf{A} := c\,\mathbf{B}.$$

Then $\mathbf{B}$ is SDD *and* PD*, and $\mathbf{A}$ is copositive (so Theorems 2.2 and 2.7 apply). Moreover*

$$\operatorname{conv}(K_1) = \operatorname{conv}(K_2) = \left[\frac{c\beta}{\beta+R}, \frac{c(\beta+R)}{\beta}\right], \quad [\mu_{\text{min}}(\mathbf{A},\mathbf{B}), \mu_{\text{max}}(\mathbf{A},\mathbf{B})] = \{c\}.$$

Hence, $[\mu_{\text{min}}(\mathbf{A},\mathbf{B}), \mu_{\text{max}}(\mathbf{A},\mathbf{B})] \subsetneq K_1 = K_2$.

Proof. Each row of $\mathbf{B}$ has diagonal entry $\beta > 0$ and the sum of the off-diagonal elements is $(n-1)\rho$, which is R by definition. Also, since $\beta > R$, then $\mathbf{B}$ is SDD and then PD. The matrix $\mathbf{A}$ is copositive as $c > 0$ and $\mathbf{B}$ is PD. Therefore Theorems 2.2 and 2.7 are valid. For the generalized spectrum, $\mathbf{B}^{-1}\mathbf{A} = c\,\mathbf{I}$, hence $\mu_{\text{min}}(\mathbf{A},\mathbf{B}) = \mu_{\text{max}}(\mathbf{A},\mathbf{B}) = c$, and therefore the interval Γ is the point $\{c\}$.

To compute K_2, note the row-sum quantities are constant across $i \in [n]$:

$$a_{ii} = c\beta, \quad r_i^+(\mathbf{A}) = cR, \quad r_i^-(\mathbf{A}) = 0, \quad b_{ii} = \beta, \quad r_i^+(\mathbf{B}) = R, \quad r_i^-(\mathbf{B}) = 0.$$

Substituting in the polynomials of Lemma 2.9 yields, for any $i \neq j$,

$$P_{ij}^{\text{low}}(y) = (y\beta - c\beta)^2 - (yR)^2 \text{ and } P_{ij}^{\text{up}}(y) = (y\beta - c\beta)^2 - (cR)^2.$$

Hence,

$$\min\{y : P_{ij}^{\text{low}}(y) = 0\} = \frac{c\beta}{\beta+R}, \quad \max\{y : P_{ij}^{\text{up}}(y) = 0\} = \frac{c(\beta+R)}{\beta},$$

independently of (i,j), so $K_2 = [c\beta/(\beta+R),\ c(\beta+R)/\beta]$. Using the formula in Corollary 2.4 gives exactly the same endpoints, hence $K_1 = K_2$.

Finally, since $\beta/(\beta+R) < 1 < (\beta+R)/\beta$ for $\beta > R > 0$, we have $\dfrac{c\beta}{\beta+R} < c < \dfrac{c(\beta+R)}{\beta}$, so $\{c\} \subsetneq K_1 = K_2$ strictly.

□

6. Discussion

We can unify the formula of K_1 in Corollary 2.4 and of K_2 in Lemma 2.9, under their common assumptions ($\mathbf{A}$ copositive and $\mathbf{B}$ SDD and PD) as follows. For $m = 1$ and $m = 2$,

$$K_m = \bigcup_{\substack{S \subset [n] \\ |S| = m}} \left[\max\{0, \min\{y \in \mathbb{R} \mid P_S^{\text{low}}(y) = 0\}\}, \max\{y \in \mathbb{R} \mid P_S^{\text{up}}(y) = 0\}\right] \tag{6.1}$$

$$\text{given} \quad P_S^{\text{up}}(y) := \prod_{i \in S}\big(b_{ii}y - a_{ii}\big) - \prod_{i \in S}\big(r_i^+(\mathbf{A}) + y\, r_i^-(\mathbf{B})\big), \tag{6.2}$$

$$P_S^{\text{low}}(y) := \prod_{i \in S}\big(a_{ii} - b_{ii}y\big) - \prod_{i \in S}\big(r_i^-(\mathbf{A}) + y\, r_i^+(\mathbf{B})\big) \quad \forall y \in \mathbb{R}. \tag{6.3}$$

In these two cases, m denotes the number of rows considered to derive each interval. One may ask if this formula extends to $m \geq 3$. Unfortunately, Example 4.3 provides a counter-example for $m = 3$. Indeed, defining (6.2) and (6.3) for the triple $S = \{1,2,3\}$ gives

$$P_S^{\text{up}}(y) = (6y-14)(10y-11)(10y-13) - 2, \quad P_S^{\text{low}}(y) = (14-6y)(11-10y)(13-10y).$$

The smallest real root of P_S^{low} is 1.1, yet the complementarity eigenvalue $\lambda \approx 0.822 \in \Pi$ lies to its left. The largest real root of P_S^{up} is ≈ 2.336, yet $\lambda \approx 2.347 \in \Pi$ lies to its right (indeed $P_S^{\text{up}}(2.347) \approx 8.5 > 0$). The question of finding a hierarchy of localization sets based on the size of the considered submatrices remains thus open.

7. Conclusions

We analyzed the symmetric EiCP$(\mathbf{A},\mathbf{B})$ with $\mathbf{B}$ being positive definite and strictly diagonally dominant. Extending the He-Liu-Shen enclosures to a more general positive definite matrix $\mathbf{B}$, we present Gershgorin-type localization sets K_1 and K_2, the latter improving upon the former, assuming $\mathbf{A}$ copositive. The bounds on these sets provide inexpensive approximations of the smallest and largest complementarity eigenvalues, as do the smallest and largest generalized eigenvalues, and no dominance exists between these bounds.

Our results also suggest several directions for future research. In an algorithmic use, the localization sets or their bounds provide fast certificates for candidates of complementarity eigenpairs. The discussion opens the question of generalizing the approach to multi-row coupling. Finally, this approach could be adapted to EiCP variants, such as the tensor eigenvalue complementarity problem.

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